Exercise 1 *The only question is how to translate propositional letters to* $1st$ *order logic. The first idea that comes to mind is to have for every propositional letter* P *a* 0*-place (nullary) predicate symbol with the same name,* P*, in our* $1st$ *order logic. Unfortunately, this is not possible for* $1st$*-order logic as presented in the book by Fitting, since predicate symbols are required to have a* positive *number of places (Definition 5.1.1).*[1] *An alternative idea then is to have a single unary predicate symbol* P *and (countably many) constants* c_i *in our* $1st$ *order logic,*[2] *and then translate the* i*th propositional letter* P_i *by the formula* $P(c_i)$*, and translate all connectives 'as themselves'. We denote this translation by* $\langle\!\langle . \rangle\!\rangle$*.*[3]

Observe that for any propositional logic formula X *its translations* $\langle\!\langle X \rangle\!\rangle$ *is a* $1st$ *order formula that is a* sentence *(*closed*; no free variables) and* quantifier-free *(no* $\forall$*,* $\exists$*). From the former it follows that its truth value does not depend on assignments:* $\langle\!\langle X \rangle\!\rangle^{I,A} = \langle\!\langle X \rangle\!\rangle^{I,A'}$ *for all interpretations* I *and assignments* A, A'*. Hence we will simply write* $\langle\!\langle X \rangle\!\rangle^{I}$*. From the latter it follows that the same holds for the subformulas, i.e. assignments will play no role at all in determining the truth value in a model, of a translated formula.*

We want to show this translation is 'good', in the sense that for every propositional formula X*,* X *is valid/satisfiable/a contradiction (Definitions 2.4.4/5) iff so is the* $1st$*-order formula* $\langle\!\langle X \rangle\!\rangle$ *(Definition 5.3.6). To that end, we first unfold the respective definitions.*

- *A propositional logic formula* X *is valid if* $v(X) = \mathbf{t}$ *for every* valuation v*, mapping the propositional letters to truth values. The* $1st$*-order logic formula* $\langle\!\langle X \rangle\!\rangle$ *is valid if* $\langle\!\langle X \rangle\!\rangle$ *is true in all models, which means (using the above observation) that for every model* $M = \langle D, I \rangle$*,* $\langle\!\langle X \rangle\!\rangle^{I} = \mathbf{t}$*.*

  Given any such $1st$*-order model* $M = \langle D, I \rangle$*, we may define the valuation* v_M *by* $v_M(P_i) = \mathbf{t}$ *if* $c_i^I \in P^I$ *and* $\mathbf{f}$ *otherwise. Then* $v_M(P_i) = \mathbf{t}$ *iff* $c_i^I \in P^I$ *iff iff* $P(c_i)^I = \mathbf{t}$ *iff* $\langle\!\langle P_i \rangle\!\rangle^I = \mathbf{t}$*. By induction this correspondence extends from propositional letters to all formulas:* $v_M(X) = \mathbf{t}$ *iff* $\langle\!\langle X \rangle\!\rangle^I = \mathbf{t}$*. Thus, if* X *is a tautology, then* $\langle\!\langle X \rangle\!\rangle$ *must be so as well.*

  Vice versa, given a valuation v*, we define the Herbrand model* $M_v = \langle D, I_v \rangle$ *by setting* $P^{I_v} = \{c_i \mid v(P_i) = \mathbf{t}\}$*, i.e. we let the predicate* P *be true on all constants* c_i *such that the corresponding propositional letter* P_i *is mapped to true by* v*. From this we obtain analogously to the above that* $v(X) = \mathbf{t}$ *iff* $\langle\!\langle X \rangle\!\rangle^I = \mathbf{t}$*. If* $\langle\!\langle X \rangle\!\rangle$ *is a tautology then certainly it is true in all Herbrand*

[1] This is non-standard (cf. for example the wikipedia page on first-order logic), but that was the reason for having this exercise at all (otherwise it simply is already part of the Ba logic course material): is it problematic not to have 0-place predicate symbols? Something similar applies to function and constant symbols: allowing function symbols to have arbitrary non-negative arity would obviate the need to treat function symbols (arity > 0) and constant symbols (arity 0) separately, as is now the case in the book (Definition 5.1.1).

[2] Instead of constants c_i, we could employ variables x_i, but then we would not translate sentence to sentences.

[3] The translation needs countably many constants. This could be reduced by switching to unary natural numbers, needing only a single function symbol S and a single unary constant 0 to represent the ith propositional letter P_i as $P(S^i(0))$.

models, hence by the correspondence X is true for all valuations, i.e. X is a tautology.

- *The correspondences for satisfiability and being a contradiction follow from that for validity in the previous item.*

  X is a contradiction iff $\neg X$ is valid iff $\langle\!\langle \neg X \rangle\!\rangle$ is valid iff $\neg\langle\!\langle X \rangle\!\rangle$ is valid iff $\langle\!\langle X \rangle\!\rangle$ is a contradiction, where we use the above correspondence for validity, and that the translation commutes with negation.

  Using that, we obtain X is satisfiable iff X is not a contradiction iff $\langle\!\langle X \rangle\!\rangle$ is not a contradiction iff $\langle\!\langle X \rangle\!\rangle$ is satisfiable.

Remark 1 *Note how the Herbrand model allows c_i to be interpreted differently from c_j for all $i \neq j$, and hence also allows that $P(c_i)$ has a truth value different from that of $P(c_j)$. The latter is essential to be able to conclude that if $\langle\!\langle X \rangle\!\rangle$ is valid then so is X. To see what could go wrong otherwise, suppose we would restrict attention to models $M = \langle D, I \rangle$ where the domain D only has, say,* one *element. Then the formula $P(c_1) = P(c_2)$ would be valid since in such models $c_1^I = c_2^I$, but the corresponding propositional logic formula $P_1 = P_2$ is obviously not valid.*

Several people took a fixed *a domain $D = \{\mathbf{t}, \mathbf{f}\}$ of* truth *values, i.e. having 'variables of type boolean', in their translation. This is incorrect in two ways:*

1. *Formulas are part of the* syntax, *not of the* semantics. *That is, when translating a formula we cannot fix D.* Validity *of a formula is truth with respect to* arbitrary *models, not a* fixed *one. That is, in translating a propositional formula X into a first-order formula X' such that X is valid iff X' is, we cannot assume anything about the domains D for the latter, other than as captured by the formula. In this way, formulas (syntax) can be seen as* characterising *(classes of) models (semantics).*[4]

2. *It's a 'type-mismatch'. Propositional letters should* not *be translated as constants in/variables ranging over the domain. Propositional letters are 'of type boolean', whereas constants are 'of type object'. Predicates, predicate properties of objects not of booleans, for example,* $\mathsf{Blue}(\mathsf{chair})$, *or* $\mathsf{Loves}(\mathsf{Abooksigun}, \mathsf{Minnie})$. *'Typing' the syntax of 1st order logic, we have for function symbols $f : D^n \to D$, for constants $c : D$, for predicates $P : D^n \to B$, and for connectives $\circ : B^n \to B$, where $B = \{\mathbf{f}, \mathbf{t}\}$.*

Exercise 2 *(5.2.2) The intuition is that substitutions only change the variables, so if two substitutions change these 'in the same way', the result will be the same.*

Formally, we show that if σ and τ are substitutions that agree on the variables of the term t, that $t\sigma = t\tau$ by induction on terms, as suggested, using

[4] There are natural properties of models that cannot be characterised by 1st-order logic formulas, such as finiteness (by compactness) or uncountability (by Löwenheim–Skolem). The other way around, if one would want to be able to express such properties, one may consider *extending* the logical language with appropriate primitives. Indeed, many such extensions exist and are used. (But cf. Lindström's theorem showing that 1st-order logic is strongest in some sense.)

Definition 5.2.2, where for σ and τ to agree *on the variables in t means that if a variable x occurs in t, then $\sigma(x) = \tau(x)$.*

- *In the base case, a term is either a variable or a constant.*

  For a variable x we conclude by the assumption that σ and τ agree on all variables in the term, so in this case on x, that: $x\sigma = \sigma(x) = \tau(x) = x\tau$.

  For a constant c we conclude since substitutions 'have no effect' on then by Definition 5.2.2 we have $c\sigma = c = c\tau$.

- *In the step case, for an arbitrary n-ary function symbol f with $n > 0$ and arbitrary terms $t_1, \ldots, t_n$, we conclude 'by substitutions acting homomorphically': $f(t_1,\ldots,t_n)\sigma = f(t_1\sigma,\ldots,t_n\sigma) = f(t_1\tau,\ldots,t_n\tau) = f(t_1\tau,\ldots,t_n)\tau$, where the first and third equalities follow by Definition 5.2.2, and the second equality by n applications of the IH showing $t_i\sigma = t_i\tau$ for $1 \leq i \leq n$, where the IH applies since if σ, τ agree on the variables in t, they certainly agree on the variables in each t_i, as the latter are a subset of the former.*

Remark 2 *Several people forgot the last part of the reasoning above, namely that the IH applies to t_i, i.e. that σ and τ agree on the variables of t_i. Although that follows trivially from the same property for t, it is essential.*

Exercise 3 *(5.2.4. Bonus) The support for a substitution comprises those variables on which the substitution 'makes a difference'. Intuitively, the composition of two substitution cannot 'make a difference' on variables other than those on which each of them individually makes a difference.*

To prove the intuition correct, we show that for all substitutions σ, τ, $\mathsf{sup}(\sigma\tau) \subseteq \mathsf{sup}(\sigma) \cup \mathsf{sup}(\tau)$, where $\mathsf{sup}(.)$ denotes the support. But this holds, since if $x \notin \mathsf{sup}(\sigma)$ and $x \notin \mathsf{sup}(\tau)$, then $x\sigma = x$ and $x\tau = \tau$, so $x(\sigma\tau) = (x\sigma)\tau = x\tau = x$, where the first equality holds by Definition 5.2.3 and the last two equalities hold by the respective assumptions, so $x \notin \mathsf{sup}(\sigma\tau)$.

From this the statement follows, since if both $\mathsf{sup}(\sigma)$ and $\mathsf{sup}(\tau)$ are finite, so is there union, hence also any subset of that union.

Remark 3 *The inclusion shown above may be proper. For instance, if $\sigma(x) = y$ and $\sigma(y) = x$ and σ maps all other variables to themselves, we have $\mathsf{sup}(\sigma\sigma) = \varnothing \subset \{x, y\} = \mathsf{sup}(\sigma) = \mathsf{sup}(\sigma) \cup \mathsf{sup}(\sigma)$ (the composition $\sigma\sigma$ is just the identity function on variables).*

Exercise 4 *(5.3.2) We show that for every model $M = \langle D, I\rangle$, every 1st order formula Φ is true in M, iff $(\forall x)\Phi$ is true in M, by showing both implications.*

For the only–if-direction, assume Φ is true in M, which means by Definition 5.3.6 that $\Phi^{I,A} = \mathbf{t}$ for all assignments A. We must show $(\forall x)\Phi$ is true in M, i.e. that $((\forall x)\Phi)^{I,A} = \mathbf{t}$ for all assignments A. By Definition 5.3.5, $((\forall x)\Phi)^{I,A} = \mathbf{t}$ iff $\Phi^{I,B} = \mathbf{t}$ for every assignment B that is an x-variant in M, i.e. for every assignment B that assigns the same values to every variable as A, except possibly x (Definition 5.3.4). But by assumption the latter holds for all *assignments, so certainly for B, from which we conclude.*

For the if-direction, the reasoning is 'in reverse'. Suppose A is an assignment. By assumption, $((\forall x)\Phi)^{I,A} = \mathbf{t}$, from which we conclude by Definition 5.3.4 since then in particular $\Phi^{I,A} = \mathbf{t}$, as A is an x-variant of itself.

Remark 4 *Many people argued intuitively here, which is in some way fine,* because *the semantics was designed to correspond to the intuition, but beware that formally we can only work with the formal semantics as given in Definitions 5.3.6 and 5.3.5. In particular, the semantics of $(\forall x)\Phi$ is defined using x-variants of assignments.*

Exercise 5 *(5.3.9) The idea for our solution is to take the simplest (mathematical)* infinite *structure we know of, the natural numbers, and try to incorporate enough of its properties so that any* finite *subset/initial segment of them will not meet them.*

The property capturing infinity of natural numbers is that every number has a successor*, which using the binary symbol R for this, may be modelled by the formula Φ_1 defined as $((\exists x)(\exists y)R(x,y)) \wedge (\forall x)(((\exists y)R(y,x)) \supset ((\exists z)R(x,z)))$; with the first conjunct capturing that there is an element with a successor (take '0' for x), and the second conjunct capturing that if an element has a predecessor it also has a successor.*

However, Φ_1 on its own allows finite models such as $\langle\{0\}, I\rangle$ with $R^I = \{\langle 0,0\rangle\}$. Of course, that is unintended; successors should be new, i.e. distinct from their predecessors. To that end, we additionally require R to be irreflexive*, by Φ_2 defined as $(\forall x)\neg R(x,x)$, excluding loops as in the example, and moreover to be* transitive*, by Φ_3 defined as $(\forall x)(\forall y)(\forall z)(R(x,y) \supset R(y,z) \supset R(x,z))$, excluding cycles of arbitrary length (when combined with irreflexivity). Then intuitively, Φ defined by $\Phi_1 \wedge \Phi_2 \wedge \Phi_3$ fits the bill, but let's formally prove it:*

- *For a proof by contradiction suppose Φ were true in a* finite *model $\langle D, I\rangle$. By the left conjunct of Φ_1 being true, then there is a pair $\langle d_0, d_1\rangle \in R^I$ with $d_0 \neq d_1$ since Φ_2 is true. We inductively define for $n \geq 2$, d_{n+2} to be an (arbitrary) element of D such that $\langle d_{n+1}, d_{n+2}\rangle \in R^I$. Such elements must exist by the right conjunct of Φ_1. Moreover, by induction on n it follows that for all $i < j \leq n+2$, $\langle d_i, d_j\rangle \in R^I$: for $j < n+2$ this holds by the IH. For $j = n+2$ and $i = n+1$ it holds per construction, and for $j = n+2$ and $i \leq n$ it follows from $\langle d_{n+1}, d_{n+2}\rangle \in R^I$ per construction, $\langle d_i, d_{n+1}\rangle \in R^I$ by the IH, and transitivity Φ_3. Thus $d_1, d_2, \ldots$ is an infinite sequence of pairwise distinct elements of D, so D is not finite.*

- *If D is infinite, then it has a countably infinite subset, say $\{d_0, d_1, d_2, \ldots\}$. Defining $R^I = \{\langle d_i, d_j\rangle \mid i < j\}$, we see Φ is true in $\langle D, I\rangle$: to make the left conjunct of Φ_1 true we may assign, say, d_0 and d_1 to x and y, and to see the right conjunct is true note that for $R(x,y)^{I,A}$ to be true in this model, A must have assigned some d_i to x, and then we can simply assign d_{i+1} to z. Then we conclude since R^I is irreflexive and transitive because $<$ is.*

Remark 5 *It is an interesting challenge*[5] *to find the* smallest *formula, for some*

[5] In fact, this challenge is offered as a Ba project.

reasonable measure of size (e.g. the number of connectives, predicate symbols, and variables in a formula), having the properties as requested in the exercise.

Other than using the property of being infinitely large *as above, one can also try to make use of being infinitely* small*; think of there being between any two real numbers another real number. This could be captured by requiring the relation to be* dense*, i.e. defining* Φ_1 *by* $(\forall x)(\forall y)(\exists z)(R(x,y) \supset (R(x,z) \wedge R(z,y)))$*; and some additional properties.*

As noted in the book (page 133), the dual of the above is impossible: by compactness, there is no sentence true in finite domains but not in infinite ones.

Several people have given some formula $\Phi = \Phi_1 \wedge \ldots \wedge P_n$ *which supposedly had the desired properties, of not having finite models and having some infinite model, but do not show these properties. Even if sometimes explanations of the components* P_i *are given, it then is not clear at all why these are needed (or could some be dropped?) to obtain both properties. For instance,* $(\forall x)(\exists y)R(x,y)$*, as answered several times, contains the right idea (there's always something next), but is on its own not quite sufficient (as discussed above in the solution). You should always prove, or at least argue, that your solution is correct.*

Exercise 6 *(5.4.1) The intuition for this is: in Herbrand models values in the domain* are *(closed) terms, so assigning values can be brought about by substitution; and once we have substituted values, their interpretation is fixed (not changed by further substitutions since values are* closed *terms).*

Formally, let $M = \langle D, I\rangle$ *be a Herbrand model for the language* L*, i.e.* D *comprises the closed terms over the function and constant symbols. We prove that for every 1st order formula* Φ*,* $\Phi^{I,A} = (\Phi A)^I$*, by structural induction on the formula, using Definitions 5.3.5 and 5.2.11 for simplifying the lhs and rhs respectively.*

- *For the atomic cases,*

  $[P(t_1,\ldots,t_n)]^{I,A} = \mathbf{t}$ *iff* $\langle t_1^{I,A},\ldots,t_n^{I,A}\rangle \in P^I$ *iff* $\langle (t_1A)^I,\ldots,(t_nA)^I\rangle \in P^I$ *iff* $P((t_1A)^I,\ldots,(t_nA)^I) = \mathbf{t}$ *iff* $(P(t_1,\ldots,t_n)A)^I = \mathbf{t}$ *, where the second iff holds by (n times) Proposition 5.4.2;*

  $\top^{I,A} = \mathbf{t} = \top^I = (\top A)^I$*; and*

  $\bot^{I,A} = \mathbf{f} = \bot^I = (\bot A)^I$*;*

- $[\neg X]^{I,A} = \neg[X^{I,A}] = \neg[(XA)^I] = [\neg(XA)]^I = [(\neg X)A]^I$*, where the second equality holds by the IH for* X*;*

- $[X \circ Y]^{I,A} = X^{I,A} \circ Y^{I,A} = (XA)^I \circ (YA)^I = [(XA)\circ(YA)]^I = [(X\circ Y)A]^I$ *where again the second equality holds by the IH for* X *and* Y*;*

- $[(\forall x)\Phi]^{I,A} = \mathbf{t}$ *iff* $\Phi^{I,B} = \mathbf{t}$ *for every assignment* B *that is an* x*-variant of* A*, iff* $[\Phi B]^I = \mathbf{t}$ *for every assignment* B *that is an* x*-variant of* A*, iff* $[\Phi A_x B]^I = \mathbf{t}$ *for every assignment* B *that is an* x*-variant of some arbitrary assignment, iff* $[\Phi A_x]^{I,B} = \mathbf{t}$ *for every assignment* B *that is an* x*-variant* *of some arbitrary assignment,* $[(\forall x)(\Phi A_x)]^I = \mathbf{t}$ *iff* $[((\forall x)\Phi)A]^I = \mathbf{t}$*, where the second iff holds by the IH for* Φ*, and the third iff by composing substitutions; and*

- *This follows as the previous item, replacing '*$\forall$*' by '*$\exists$*', and 'every' by 'some'.*

Exercise 7 *(5.4.2) Suppose* ϕ *is a formula of* L *and* $M = \langle D, I\rangle$ *is a Herbrand model for* L*. Then:*

- *To show* $(\forall x)\Phi$ *is true in* M *iff* $\Phi\{x/d\}$ *is true in* M *for every* $d \in D$*.*

  Unfolding Definitions 5.3.6 and 5.3.5 and using Proposition 5.4.3, the lhs is equivalent to $[\Phi B]^I = \mathbf{t}$ *for every assignment* A *and every assignment* B *that is an* x*-variant of* A*, and the rhs to* $[\Phi\{x/d\}A]^I$ *for every* $d \in D$ *and every assignment* A*. From this we conclude, since an assignment that is an* x*-variant of* A is *a composition of singleton assignment* $\{x/d\}$ *and* A*;*

- *To show* $(\forall x)\Phi$ *is true in* M *iff* $\Phi\{x/d\}$ *is true in* M *for some* $d \in D$*.*

  This follows as in the previous item, replacing '$\forall$*' by '*$\exists$*', and 'every' by 'some'.*

Exercise 8 *(5.5.2, Bonus) Suppose* L *is a first-order language and* $M = \langle D, I\rangle$ *is a Herbrand model for* L*.*

Below in each case, we proceed by unfolding Definitions 5.3.5 and 5.3.6, using Proposition 5.4.3 and distinguishing case according to Table 5.1.

- *To show: if* γ *is a formula of* L*,* γ *is true in* M *iff* $\gamma(d)$ *is true in* M *for every* $d \in D$*.*

  By Definition 5.3.6 we must show $\gamma^{I,A}$ *is true for all substitutions* A *iff* $\gamma(d)^{I,A}$ *is true for all substitutions* A*. We distinguish cases on the (two) possible shapes of* γ *in Table 5.1.*

  If $\gamma = (\forall x)\Phi$*, then* $\gamma(d) = \Phi\{x/d\}$*, and proceeding as stated above both sides are seen to be equivalent to* $[\Phi\{x/d\}A]^I$ *being true for all substitutions* A *and every* d*.*

  If $\gamma = \neg(\exists x)\Phi$*, then* $\gamma(d) = \neg\Phi\{x/d\}$*, then both sides are equivalent to* $[\neg\Phi\{x/d\}A]^I$ *being true for all substitutions* A *and every* d*.*

- *To show: if* δ *is a formula of* L*,* δ *is true in* M *iff* $\delta(d)$ *is true in* M *for some* $d \in D$*.*

  This follows as in the previous item, replacing 'γ*' by '*δ*', '*$\forall$*' by '*$\exists$*' (and correspondingly using the right instead of the left column of Table 5.1), and 'every' by 'some'.*

Exercise 9 *(5.6.3) Checking the cases of First-Order Structural Induction, we see the remaining cases are the propositional cases and the* δ*-case of the quantifiers.*

Of the propositional cases it is stated on p. 128 that they are essentially as they were earlier, that is, as established in Exercise 3.5.2. Indeed they are:

- *Suppose* $\neg\neg X$ *is a sentence of L, the result is known for simpler sentences of L, and* $\neg\neg X \in H$. *Then by the definition of Hintikka set (Definitions 5.6.1 and 3.5.1),* $X \in H$, *hence by the IH* X *is true, so by Definitions 5.3.5 and 5.36* $\neg\neg X$ *is true, since* $X^{I,A} = [\neg\neg X]^{I,A}$ *for all A.*

- *Suppose* α *is a sentence of L, the result is known for simpler sentences of L, and* $\alpha \in H$. *Then by definition of Hintikka set (Definitions 5.6.1 and 3.5.1),* $\alpha_1, \alpha_2 \in H$, *hence by the IH* α_1 *and* α_2 *are true, hence by Definitions 5.3.5 and 5.3.6* α *is true, since* $\alpha^{I,A} = [\alpha_1 \wedge \alpha_2]^{I,A}$; *cf. Proposition 2.6.1.*

- *Suppose* β *is a sentence of L, the result is known for simpler setences of L, and* $\beta \in H$. *Then be definition of Hintikka set (Definitions 5.6.1 and 3.5.1),* $\beta_i \in H$ *for some* $i \in 1, 2$, *hence by the IH* β_i *is true for that* i, *so by Definitions 5.3.5 and 5.3.6* β *is true, since* $\beta^{I,A} = [\beta_1 \wedge \beta_2]^{I,A}$; *cf. Proposition 2.6.1.*

For the δ*-case we take the proof in the book for the* γ*-case, replacing '*γ*' by '*δ*' and 'every' by 'some': Suppose* δ *is a sentence of* l, *the result is known for simpler sentences of L, and* $\delta \in H$; *we show* δ *is true in* M. *Since* $\delta \in H$, *we have* $\delta(t) \in H$ *for some closed term* t, *since* H *is a Hintikka set. By the induction hypothesis, and the fact that* D *is exactly the set of closed terms,* $\delta(t)$ *is true in* M *for some* $t \in D$. *Then* δ *is true in* M *by (item 2 instead of item 1 of) Proposition 5.5.2*

Exercise 10 *(5.9.2) The statement derives a property for* arbitrary*, possibly* infinite*, structures from the same property for* finite *structures, so makes one think of compactness, Theorem 5.9.1.*

To enable using compacntess, we let each vertex v *of a graph* G *be represented by a constant of the same name in our language, and adjoin for every undirected edge* $\{v, w\}$ *of* G *a formula* $\{E(v, w)\}$ *to our set* S *of sentences, with the intuition that finite subsets of* S *correspond to finite subgraphs (see below). Finally, we adjoin the sentence* $\Phi = \Phi_1 \wedge \Phi_2$ *expressing* four colorability *to* S, *where* Φ_1 *is* $(\forall x)(R(x) \vee G(x) \vee B(x) \vee Y(x))$ *and* Φ_2 *is* $(\forall x)(\forall y)\{E(x,y) \supset [(R(x) \supset \neg R(y)) \wedge (G(x) \supset \neg G(y)) \wedge (B(x) \supset \neg B(y)) \wedge (Y(x) \supset \neg Y(y))]\}$.

Since by assumption every finite subgraph of G *is four colorable, every finite subset* S' *of* S *is satisfiable, namely in the subgraph* G' *comprising the vertices mentioned in formulas of* S' *and the edges between them. By compactness therefore* S *itself is satisfiable, i.e. there is a model* $\langle D, I\rangle$ *in which all sentences of* S *are true. From that, a four-coloring of* G *is obtained by assigning color R/G/B/Y to a vertex* v *if* $v^I \in R^I/G^I/B^I, Y^I$ *in the model.*

Remark 6 *Note that not every pair* (V', E') *with* $V \subseteq V'$ *and* $E \subseteq E'$ *is a* subgraph *of* (V, E) *as defined in the exercise: not only can we only have an* *edge in E' if both its end-points are in V', also the converse holds: if the end-points of an edge in E are in V', then the edge is in E'. This is the reason for working with the subgraph G' in the above solution.*

Exercise 11 *(5.9.3 Bonus) To see that a set S of sentences of L is satisfiable if and only if it is satisfiable in a model that is Herbrand with respect to L^{par}, note that the if-direction is trivial, and the only–if-direction was was shown in the first part of the proof of the Löwenheim–Skolem theorem 5.9.3 on pages 133, 134, based on the First-Order Model Existence theorem 5.8.2.*

To see that a sentence X of L is valid iff X is true in all models that are Herbrand with respect to L^{par}, we reason using the above as follows X is valid iff $\neg X$ is not satisfiable iff $\neg X$ is not satisfiable in a model that is Herbrand with respect to L^{par} iff X is true in all models that are Herbrand with respect to L^{par}.

Exercise 12 *(5.10.1) Using the notation from Definition 5.10.1 to state the second item, we have to show X is true in every model in which the members of S are true iff $\forall X$ is true in every model in which the members of $\forall S$ are true. This follows by repeatedly applying Exercise 5.3.2, stating that a formula Φ is true in a model iff $(\forall x)\Phi$ is, to the formulas in S and X.*

Remark 7 *Note that this exercise employs the first reading of what it could mean for a* formula *X to be a logical consequence of a set S of* formulas *(where the formulas need not be closed, i.e. need not be sentences), as discussed on page 135, allowing to assign* distinct *values to the free variables of distinct formulas. See that page and Exercise 5.10.2 for the other reasonable reading, namely assigning the* same *values to free variables occurring in distinct formulas (which can be reduced to the notion of logical consequences for sentences as well, by means of* substituting parameters for variables *instead of by universal quantification as above).*

Exercise 13 *(5.10.3)*

- *Unfolding the definitions of validity in the lhs according to Definition 5.3.6 and of logical consequence in the rhs according to Definition 5.10.1, we must show that X is true in every model iff X is true in every model in which all the members of S are true, for every set S. The only–if-direction being trivial, the if-direction follows by instantiating S with the empty set.*
- *That $S \vDash_f X$ for every X, if $A \in S$ and $\neg A \in S$ holds vacuously, since there are no models in which both A and $\neg A$ are true.*