Computational Logic	SS 2020	703824 VU

TAKE HOME EXAM	May 12, 9:00 – May 13, 9:00, 2020[1]

> There are 42 points available, plus 4 bonus points. The points scored are added to the crosses of the assignments and divided by 0.82 to determine your percentage for the course. You need at least 50 percent to pass.

$\boxed{1}$ This exercise is about first-order logic. Consider the first-order formula $\varphi = (\forall x)(\exists y)[P(x) \supset Q(y)] \supset (\forall x)[P(a) \supset (\exists y)Q(y)]$ with a a constant. It is a tautology.

[4] (a) First stepwise compute a prenex form φ' of φ, in each step giving the quantifier rewrite rule employed, and next Skolemize φ', per quantifier, to obtain a formula φ''.

[3] (b) Consider the singleton set $S = \{\varphi\}$. Give a Herbrand model of S (Herbrand with respect to the original first-order language of φ extended with parameters).

[4] (c) Give a tableau proof of φ.

[12] $\boxed{2}$ This exercise is about proofs. Answer $\boxed{\textbf{three}}$ of the following five items (4 points per item).

(a) Prove that if X and Y are propositional formulas that are *equivalent*, i.e. $v(X \equiv Y) = \mathbf{t}$ for all valuations v, then their duals X^d and Y^d are equivalent too. Illustrate this for $X = P \vee (\neg Q \wedge R)$ and $Y = (P \vee \neg Q) \wedge (P \vee R)$.

(b) Suppose we change the 4th clause of the definition of propositional Hintikka set (Definition 3.5.1) in the following way:

4. $\alpha \in \mathbf{H} \;\Rightarrow\; \alpha_1 \in \mathbf{H}$

giving rise (combined with the other, unchanged, clauses) to what we will call *Hintikka'* sets. The notion of Hintikka' set is not a good one. Give a relevant property (for showing completeness results) that Hintikka sets have, and show (how) it fails for Hintikka' sets.

(c) Transform the following tableau into a *cut-free* tableau using the cut-elimination procedure from the lecture/book:

$$\neg(((A \supset B) \wedge (\neg A \supset B)) \supset B)$$

$$\begin{array}{cc}
A & \neg A \\
(A \supset B) \wedge (\neg A \supset B) & (A \supset B) \wedge (\neg A \supset B) \\
\neg B & \neg B \\
A \supset B & A \supset B \\
\neg A \supset B & \neg A \supset B \\
\neg A \quad\quad B & \neg\neg A \quad\quad B
\end{array}$$

[1] Times in CEST

(d) Consider the propositional formulas $\varphi = \neg P \supset (P \supset \bot)$ and $\varphi' = (P \supset \bot) \supset \neg P$. Give proofs of φ and φ' using the 9 Axiom Schemes and MP of the Hilbert System of Fitting. (Recall that in Fitting, $\neg P$ is *not* an abbreviation of $P \supset \bot$.)

(e) Prove that for *maximal* states in a Kripke model (intuitionistic) forcing coincides with (classical) truth. Formally, let some Kripke mode be given and let c be *maximal* in it, i.e. for all c', if $c \leqslant c'$ then $c = c'$. Prove that if we define for all propositional letters $v(p) = \mathbf{t}$ if $c \Vdash p$, then for all propositional formulas ϕ constructed from propositional letters and $\supset, \bot, \vee, \wedge$, we have $v(\phi) = \mathbf{t}$ iff $c \Vdash \phi$. Illustrate this for the formula $((p \supset \bot) \supset \bot) \supset p$.

$\boxed{3}$ This exercise is about the Curry–Howard isomorphism. Let B be the λ-term $\lambda f\, g\, x. f\,(g\, x)$

[3] (a) Give the ND proof (in tree form) of $(\psi \supset \chi) \supset (\phi \supset \psi) \supset (\phi \supset \chi)$ for all propositional formulas ϕ, ψ, χ, corresponding to the λ-term B.

[4] (b) Show that, for all propositional formulas ϕ, $\phi \supset \phi$ can be proven indirectly by using B. More precisely, first show that $\vdash B\,(\lambda x.x)\,(\lambda x.x) : \tau \to \tau$ for all simple types τ and that that gives an ND proof that $\phi \supset \phi$ for all propositional formulas ϕ. Next, show that $B\,(\lambda x.x)\,(\lambda x.x)$ normalizes to $\lambda x.x$ (possibly up to renaming) by repeated uses of $\to_\beta$.

[4] (c) The λ-term $\lambda x.x$ both behaves as the identity and has the type of the identity. Formally, both $(\lambda x.x)\, M \to_\beta^* M$ for every λ-term M and $\vdash \lambda x.x : \tau \to \tau$ for every simple type τ. Does every (closed) λ-term that has the type of the identity, behave as the identity? That is, prove or give a counterexample to that for every simply typed λ-term E, if $\vdash E : \tau \to \tau$ for every simple type τ, then $E\, M \to_\beta^* M$ for all λ-terms M.

[4] (bonus) The CL-term $\mathsf{S\,K\,K}$ both behaves as the identity and has the type of the identity. Formally, both $\mathsf{S\,K\,K}\, M \to_w^* M$ for every CL-term M and $\vdash \mathsf{S\,K\,K} : \tau \to \tau$ for every simple type τ. Does every CL-term that behaves as the identity have the type of the identity? That is, prove or give a counterexample to that for every simply typed CL-term E, if $E\, M \to_w^* M$ for all CL-terms M, then $\vdash E : \tau \to \tau$ for every simple type τ.

[8] $\boxed{4}$ Determine whether the following statements are true or false.

Every correct answer is worth 2 points. For every wrong answer 1 point is subtracted, provided the total number of points is non-negative.

(a) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are propositional consistency properties, then their intersection $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$ is a propositional consistency property again.

(b) In propositional logic, if Z, Z' are both interpolants of $X \supset Y$, then $Z \supset Z'$ is a tautology or $Z' \supset Z$ is a tautology (or both).

(c) If ϕ_1 and ϕ_2 are obtained by Skolemising (possibly distinct) prenex forms of the same first-order formula ϕ, then ϕ_1 and ϕ_2 have the same number of constants.

(d) The formula $(\exists x)R(x, f(b)) \wedge (\exists y)R(a, y)$ is an interpolant of

$$(R(a, b) \wedge (\forall x)(\exists y)(R(a, x) \supset R(y, f(x)))) \supset (R(a, c) \vee (\exists x)R(x, f(b)))$$