Computational Logic

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Outline

- Summary of Previous Lecture
- Model Existence Theorem
- Compactness
- Interpolation
- Semantic Tableaux
- Hilbert Systems
- Exercises
- Further Reading

Tableau Expansion Rules

$$\frac{\neg\neg Z}{Z} \qquad \frac{\neg\bot}{\top} \qquad \frac{\neg\top}{\bot} \qquad \frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta}{\beta_1 \mid \beta_2}$$

Definition

finite set $\{A_1, \dots, A_n\}$ of propositional formulas

1. following one-branch tree is tableau for $\{A_1, \dots, A_n\}$:

$$\begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_n \end{array}$$

2. if T is tableau for $\{A_1, \dots, A_n\}$ and T^* results from T by application of tableau expansion rule then T^* is tableau for $\{A_1, \dots, A_n\}$

Definitions

- branch θ of tableau is closed if both X and $\neg X$ occur on θ for some propositional formula X, or if $\bot$ occurs on θ
- branch θ of tableau is atomically closed if both A and $\neg A$ occur on θ for some propositional letter A, or if $\bot$ occurs on θ
- tableau is (atomically) closed if every branch is (atomically) closed
- tableau proof of X is closed tableau for $\{\neg X\}$
- X is theorem if X has tableau proof, denoted by $\vdash_{pt} X$
- tableau is strict if no formula has had Tableau Expansion Rule applied to it twice on same branch
- tableau branch θ is satisfiable if set of propositional formulas on it is satisfiable
- tableau T is satisfiable if at least one branch of T is satisfiable

Lemma

any application of Tableau Expansion Rule to satisfiable tableau yields another satisfiable tableau

Lemma

if S admits closed tableau then S is not satisfiable

Theorem (Propositional Tableau Soundness)

if X has tableau proof then X is tautology

Definition

set **H** of propositional formulas is propositional Hintikka set provided

1. for any propositional letter A, not both $A \in \mathbf{H}$ and $\neg A \in \mathbf{H}$
2. $\bot \notin \mathbf{H}$, $\neg\top \notin \mathbf{H}$
3. if $\neg\neg Z \in \mathbf{H}$ then $Z \in \mathbf{H}$
4. if $\alpha \in \mathbf{H}$ then $\alpha_1 \in \mathbf{H}$ and $\alpha_2 \in \mathbf{H}$
5. if $\beta \in \mathbf{H}$ then $\beta_1 \in \mathbf{H}$ or $\beta_2 \in \mathbf{H}$

Lemma (Hintikka's Lemma)

every propositional Hintikka set is satisfiable

Definition

collection $\mathcal{C}$ of sets of propositional formulas is propositional consistency property if, for each $S \in \mathcal{C}$:

1. for any propositional letter A, not both $A \in S$ and $\neg A \in S$
2. $\bot \notin S$, $\neg\top \notin S$
3. if $\neg\neg Z \in S$ then $S \cup \{Z\} \in \mathcal{C}$
4. if $\alpha \in S$ then $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$
5. if $\beta \in S$ then $S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$

if $\mathcal{C}$ is propositional consistency property then $S \in \mathcal{C}$ is called $\mathcal{C}$-consistent

Theorem (Propositional Model Existence)

if $\mathcal{C}$ is propositional consistency property and $S \in \mathcal{C}$ then S is satisfiable

Part I: Propositional Logic

compactness, completeness, Hilbert systems, Hintikka's lemma, interpolation, logical consequence, model existence theorem, propositional semantic tableaux, soundness

Part II: First-Order Logic

compactness, completeness, Craig's interpolation theorem, cut elimination, first-order semantic tableaux, Herbrand models, Herbrand's theorem, Hilbert systems, Hintikka's lemma, Löwenheim-Skolem, logical consequence, model existence theorem, prenex form, skolemization, soundness

Part III: Limitations and Extensions of First-Order Logic

Curry-Howard isomorphism, intuitionistic logic, Kripke models, second-order logic, simply-typed λ-calculus

Outline

Definition

collection $\mathcal{C}$ of sets of propositional formulas is propositional consistency property if, for each $S \in \mathcal{C}$:

1. for any propositional letter A, not both $A \in S$ and $\neg A \in S$
2. $\bot \notin S$, $\neg\top \notin S$
3. if $\neg\neg Z \in S$ then $S \cup \{Z\} \in \mathcal{C}$
4. if $\alpha \in S$ then $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$
5. if $\beta \in S$ then $S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$

if $\mathcal{C}$ is propositional consistency property then $S \in \mathcal{C}$ is called $\mathcal{C}$-consistent

Theorem (Propositional Model Existence)

if $\mathcal{C}$ is propositional consistency property and $S \in \mathcal{C}$ then S is satisfiable

Outline

Theorem (Propositional Compactness)

if every finite subset of set S of propositional formulas is satisfiable then S is satisfiable

Proof

- let $\mathcal{C} = \{W \mid \text{every finite subset of } W \text{ is satisfiable}\}$
- $S \in \mathcal{C}$ and $\mathcal{C}$ is propositional consistency property:
  1. if $A \in W$ and $\neg A \in W$ then $W \notin \mathcal{C}$
  2. if $\bot \in W$ or $\neg\top \in W$ then $W \notin \mathcal{C}$
  3. suppose $\neg\neg Z \in W \in \mathcal{C}$ and let V be finite subset of $W \cup \{Z\}$

     $(V \cap W) \cup \{\neg\neg Z\}$ is finite subset of W and thus satisfiable

     $(V \cap W) \cup \{\neg\neg Z, Z\}$ is satisfiable

     $V \subseteq (V \cap W) \cup \{\neg\neg Z, Z\}$ is satisfiable

Proof (cont'd)

- let $\mathcal{C} = \{W \mid \text{every finite subset of } W \text{ is satisfiable}\}$
- $S \in \mathcal{C}$ and $\mathcal{C}$ is propositional consistency property:

  4. suppose $\alpha \in W \in \mathcal{C}$ and let V be finite subset of $W \cup \{\alpha_1, \alpha_2\}$

     $(V \cap W) \cup \{\alpha\}$ is finite subset of W and thus satisfiable

     $(V \cap W) \cup \{\alpha, \alpha_1, \alpha_2\}$ is satisfiable

     $V \subseteq (V \cap W) \cup \{\alpha, \alpha_1, \alpha_2\}$ is satisfiable

  5. suppose $\beta \in W \in \mathcal{C}$

     suppose neither $W \cup \{\beta_1\}$ nor $W \cup \{\beta_2\}$ belongs to $\mathcal{C}$

     $\exists$ finite unsatisfiable subsets $F_1 \subseteq W \cup \{\beta_1\}$ and $F_2 \subseteq W \cup \{\beta_2\}$

     $(F_1 \cup F_2) \cap W \cup \{\beta\}$ is finite subset of W and thus satisfiable

     $(F_1 \cup F_2) \cap W \cup \{\beta, \beta_1\}$ or $(F_1 \cup F_2) \cap W \cup \{\beta, \beta_2\}$ is satisfiable

     $F_1 \subseteq (F_1 \cup F_2) \cap W \cup \{\beta, \beta_1\}$ and $F_2 \subseteq (F_1 \cup F_2) \cap W \cup \{\beta, \beta_2\}$ $\lightning$

Outline

Definition

formula Z is interpolant for implication $X \supset Y$ if every propositional letter of Z occurs in both X and Y, and $X \supset Z$ and $Z \supset Y$ are both tautologies

Examples

- $P \vee Q$ is interpolant for $(P \vee (Q \wedge R)) \supset (P \vee \neg\neg Q)$
- $\bot$ is interpolant for $(P \wedge \neg P) \supset Q$

Theorem (Craig Interpolation)

every tautology $X \supset Y$ has interpolant

Notation

$\langle S \rangle$ denotes conjunction of all members of finite set S of formulas

Definition

finite set S of formulas is Craig consistent if $\langle S_1 \rangle \supset \neg \langle S_2 \rangle$ has no interpolant for some partition $S_1 \uplus S_2$ of S

Lemma

collection of all Craig consistent sets is propositional consistency property

Proof

- let $\mathcal{C}$ be collection of all Craig consistent sets
- let $S \in \mathcal{C}$ so $\langle S_1 \rangle \supset \neg \langle S_2 \rangle$ has no interpolant for some partition $S_1 \uplus S_2$ of S (terminology: $S_1 \uplus S_2$ has no interpolant)

Proof (cont'd)

given $S \in \mathcal{C}$ and partition $S_1 \uplus S_2$ of S without interpolant

1. suppose $A, \neg A \in S$
   - if $A, \neg A \in S_1$ then $\bot$ is interpolant of $S_1 \uplus S_2$ ↯
   - if $A, \neg A \in S_2$ then $\top$ is interpolant of $S_1 \uplus S_2$ ↯
   - if $A \in S_1$ and $\neg A \in S_2$ then A is interpolant of $S_1 \uplus S_2$ ↯
   - if $\neg A \in S_1$ and $A \in S_2$ then $\neg A$ is interpolant of $S_1 \uplus S_2$ ↯
2. suppose $\bot \in S$
   - if $\bot \in S_1$ then $\bot$ is interpolant of $S_1 \uplus S_2$ ↯
   - if $\bot \in S_2$ then $\top$ is interpolant of $S_1 \uplus S_2$ ↯

   suppose $\neg\top \in S$
   - if $\neg\top \in S_1$ then $\bot$ is interpolant of $S_1 \uplus S_2$ ↯
   - if $\neg\top \in S_2$ then $\top$ is interpolant of $S_1 \uplus S_2$ ↯

Proof (cont'd)

given $S \in \mathcal{C}$ and partition $S_1 \uplus S_2$ of S without interpolant

3. suppose $\neg\neg Z \in S$
   - if $\neg\neg Z \in S_1$ then $(S_1 \cup \{Z\}) \uplus S_2$ has no interpolant
   - if $\neg\neg Z \in S_2$ then $S_1 \uplus (S_2 \cup \{Z\})$ has no interpolant

   hence $S \cup \{Z\} \in \mathcal{C}$
4. suppose $\alpha \in S$
   - if $\alpha \in S_1$ then $(S_1 \cup \{\alpha_1, \alpha_2\}) \uplus S_2$ has no interpolant
   - if $\alpha \in S_2$ then $S_1 \uplus (S_2 \cup \{\alpha_1, \alpha_2\})$ has no interpolant

   hence $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$

Proof (cont'd)

given $S \in \mathcal{C}$ and partition $S_1 \uplus S_2$ of S without interpolant

5. suppose $\beta \in S$ and neither $S \cup \{\beta_1\}$ nor $S \cup \{\beta_2\}$ belongs to $\mathcal{C}$
   - if $\beta \in S_1$ then $\langle S_1 \rangle \equiv \langle S_1 \cup \{\beta_1\} \rangle \vee \langle S_1 \cup \{\beta_2\} \rangle$

     $(S_1 \cup \{\beta_1\}) \uplus S_2$ is partition of $S \cup \{\beta_1\}$ and thus has interpolant γ_1

     $(S_1 \cup \{\beta_2\}) \uplus S_2$ is partition of $S \cup \{\beta_2\}$ and thus has interpolant γ_2

$$\begin{array}{ll} \langle S_1 \cup \{\beta_1\} \rangle \supset \gamma_1 & \gamma_1 \supset \neg \langle S_2 \rangle \\ \langle S_1 \cup \{\beta_2\} \rangle \supset \gamma_2 & \gamma_2 \supset \neg \langle S_2 \rangle \end{array}$$

     hence $\gamma_1 \vee \gamma_2$ is interpolant of $S_1 \uplus S_2$ $\lightning$

$$\langle S_1 \rangle \equiv \langle S_1 \cup \{\beta_1\} \rangle \vee \langle S_1 \cup \{\beta_2\} \rangle \supset \gamma_1 \vee \gamma_2 \supset \neg \langle S_2 \rangle$$

Proof (cont'd)

given $S \in \mathcal{C}$ and partition $S_1 \uplus S_2$ of S without interpolant

5. suppose $\beta \in S$ and neither $S \cup \{\beta_1\}$ nor $S \cup \{\beta_2\}$ belongs to $\mathcal{C}$
   - if $\beta \in S_2$ then $\neg\langle S_2\rangle \equiv \neg\langle S_2 \cup \{\beta_1\}\rangle \wedge \neg\langle S_2 \cup \{\beta_2\}\rangle$

     $S_1 \uplus (S_2 \cup \{\beta_1\})$ is partition of $S \cup \{\beta_1\}$ and thus has interpolant δ_1

     $S_1 \uplus (S_2 \cup \{\beta_2\})$ is partition of $S \cup \{\beta_2\}$ and thus has interpolant δ_2

$$\langle S_1\rangle \supset \delta_1 \qquad \delta_1 \supset \neg\langle S_2 \cup \{\beta_1\}\rangle$$
$$\langle S_1\rangle \supset \delta_2 \qquad \delta_2 \supset \neg\langle S_2 \cup \{\beta_2\}\rangle$$

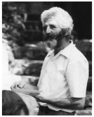
hence $\delta_1 \wedge \delta_2$ is interpolant of $S_1 \uplus S_2$ ↯

$$\langle S_1\rangle \supset \delta_1 \wedge \delta_2 \supset \neg\langle S_2 \cup \{\beta_1\}\rangle \wedge \neg\langle S_2 \cup \{\beta_2\}\rangle \equiv \neg\langle S_2\rangle$$

$\mathcal{C}$ is propositional consistency property

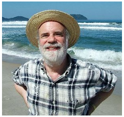
Proof (of Craig Interpolation Theorem)

- suppose $X \supset Y$ has no interpolant
- let $S = \{X, \neg Y\}$ with partition $S_1 = \{X\}$ and $S_2 = \{\neg Y\}$
- interpolant for $\langle S_1 \rangle \supset \neg \langle S_2 \rangle$ is interpolant for $X \supset Y$
  and hence does not exist
- S is Craig consistent
- S is satisfiable by Model Existence Theorem and previous lemma
- hence $X \supset Y$ is no tautology



William Craig
(1918 – 2016)

Jaakko Hintikka
(1929 – 2015)



Outline

Definition

finite set S of propositional formulas is tableau consistent if there is no closed tableau for S

Lemma

collection of all tableau consistent sets is propositional consistency property

Proof

- properties 1, 2, 3: . . . blackboard . . .
- properties 4, 5: next two slides

Proof (cont'd)

- property 4: let $\alpha \in S$ and consider $S \cup \{\alpha_1, \alpha_2\}$

  suppose $S \cup \{\alpha_1, \alpha_2\}$ is not tableau consistent

  let $S = \{\alpha, X_1, \ldots, X_n\}$

  closed tableau for $S \cup \{\alpha_1, \alpha_2\}$:

α	
X_1	
$\vdots$	
X_n	
α_1	apply α-rule
α_2	apply α-rule
rest of tableau	

Proof (cont'd)

- property 5: let $\beta \in S$ and consider $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$

  suppose neither $S \cup \{\beta_1\}$ nor $S \cup \{\beta_2\}$ is tableau consistent

  let $S = \{\beta, X_1, \ldots, X_n\}$

  closed tableaux for $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$:

β	β
X_1	X_1
$\vdots$	$\vdots$
X_n	X_n
β_1	β_2
T_1	T_2

  can be merged into closed tableau for S

Theorem (Completeness for Propositional Tableaux)

every tautology has tableau proof

Proof

- suppose formula X does not have tableau proof
- there is no closed tableau for $\{\neg X\}$
- $\{\neg X\}$ is tableau consistent
- $\{\neg X\}$ is satisfiable by Propositional Model Existence Theorem
- X cannot be tautology

Outline

- Summary of Previous Lecture
- Model Existence Theorem
- Compactness
- Interpolation
- Semantic Tableaux
  - Completeness
  - Completeness with Restrictions
  - Propositional Consequence
- Hilbert Systems
- Exercises
- Further Reading

Theorem

for every tautology X

strict tableau construction process for $\{\neg X\}$ that is continued until every non-literal formula occurrence on every branch has been used must terminate and do so in atomically closed tableau

Proof

- termination follows by considering $\sum \sum \{r(Y) \mid Y \text{ is unused formula}\}$
- suppose final tableau T is not atomically closed
- let θ be branch of T that is not atomically closed
- if $\neg\neg Z$ occurs on θ then Z occurs on θ
  if α occurs on θ then α_1 and α_2 occur on θ
  if β occurs on θ then β_1 or β_2 occurs on θ
- set of formulas S occurring on θ is Hintikka set and thus satisfiable
- $\neg X \in S$ and thus $v(\neg X) = \mathrm{t}$ for some valuation v $\lightning$

Corollary

tableau systems provide decision procedure for being tautology

Outline

Definition

propositional formula X is propositional consequence of set S of propositional formulas, denoted by $S \vDash_p X$, if X evaluates to t for every valuation v that maps every member of S to t

Remark

X is tautology if and only if $\varnothing \vDash_p X$ (simplified notation: $\vDash_p X$)

Theorem

$S \vDash_p X$ if and only if $S_0 \vDash_p X$ for some finite subset S_0 of S

Theorem

$S \vDash_p X$ if and only if $S_0 \vDash_p X$ for some finite subset S_0 of S

Proof

$\Rightarrow$ if $S \vDash_p X$ then $S \cup \{\neg X\}$ is not satisfiable

some finite subset S' of $S \cup \{\neg X\}$ is not satisfiable by compactness

let $S_0 = S' \cap S$

S_0 is finite subset of S and $S_0 \cup \{\neg X\}$ is not satisfiable

$S_0 \vDash_p X$

$\Leftarrow$ obvious

set of formulas S

Definitions

- S-introduction rule for tableaux: any member of S can be added to end of any tableau branch
- $S \vdash_{pt} X$ if there exists closed propositional tableau for $\{\neg X\}$, allowing S-introduction rule

Definitions

- tableau branch θ is S-satisfiable if union of S and set of propositional formulas on θ is satisfiable
- tableau T is S-satisfiable if at least one branch of T is S-satisfiable

Lemmata

- *any application of Tableau Expansion Rule as well as S-introduction rule to S-satisfiable tableau yields another S-satisfiable tableau*
- *there are no closed S-satisfiable tableaux*

Definition

S is $X-$tableau consistent if $S \vdash_{pt} X$ does not hold

Lemmata

for each formula X

- *collection of $X-$tableau consistent sets is propositional consistency property*
- *if S is $X-$tableau consistent then $S \cup \{\neg X\}$ is $X-$tableau consistent*

Theorem (Strong Soundness and Completeness)

for any set S of propositional formulas and any propositional formula X

$$S \vDash_p X \iff S \vdash_{pt} X$$

Proof

$\Rightarrow$ suppose $S \vdash_{pt} X$ does not hold, so S is $X-$tableau consistent

$S \cup \{\neg X\}$ is $X-$tableau consistent

$S \cup \{\neg X\}$ is satisfiable by Model Existence Theorem

$S \vDash_p X$ does not hold

$\Leftarrow$ there exists closed tableau for $\{\neg X\}$, allowing S-introduction rule

initial tableau cannot be S-satisfiable

$S \cup \{\neg X\}$ is not satisfiable

$S \vDash_p X$

Outline

Definitions

- derivation in Hilbert system from set S of formulas is finite sequence $X_1, X_2, \ldots, X_n$ of formulas such that each formula is axiom, or member of S, or follows from earlier formulas by rule of inference
- proof in Hilbert system is derivation from $\varnothing$

Definitions

given Hilbert system h

- X is consequence of set S in h, denoted by $S \vdash_{ph} X$, if X is last line of derivation from S
- formula X is theorem of h, denoted by $\vdash_{ph} X$, if X is consequence of $\varnothing$ in h

Definition (Modus Ponens)

$$\frac{X \qquad X \supset Y}{Y}$$

Definition (Axiom Scheme 1)

$$X \supset (Y \supset X)$$

Definition (Axiom Scheme 2)

$$(X \supset (Y \supset Z)) \supset ((X \supset Y) \supset (X \supset Z))$$

Example

$P \supset P$ is theorem:

1.	$(P \supset ((P \supset P) \supset P)) \supset ((P \supset (P \supset P)) \supset (P \supset P))$	Axiom Scheme 2
2.	$P \supset ((P \supset P) \supset P)$	Axiom Scheme 1
3.	$(P \supset (P \supset P)) \supset (P \supset P)$	Modus Ponens
4.	$P \supset (P \supset P)$	Axiom Scheme 1
5.	$P \supset P$	Modus Ponens

Theorem (Deduction Theorem)

in any Hilbert System h with Modus Ponens as only rule of inference and at least Axiom Schemes 1 and 2:

$$S \cup \{X\} \vdash_{ph} Y \quad \Longleftrightarrow \quad S \vdash_{ph} X \supset Y$$

Example

$(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$ is theorem:

- $\{P \supset (Q \supset R), Q, P\} \vdash_{ph} R$:

1.	$P \supset (Q \supset R)$	
2.	P	
3.	$Q \supset R$	Modus Ponens
4.	Q	
5.	R	Modus Ponens

- $\{P \supset (Q \supset R), Q\} \vdash_{ph} P \supset R$ by Deduction Theorem
- $\{P \supset (Q \supset R)\} \vdash_{ph} Q \supset (P \supset R)$ by Deduction Theorem
- $\vdash_{ph} (P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$ by Deduction Theorem

Proof (if direction)

- suppose $S \cup \{X\} \vdash_{ph} Y$
- let $\Pi_1 \colon Z_1, \dots, Z_n$ be derivation of Y from $S \cup \{X\}$, so $Z_n = Y$
- consider new sequence $\Pi_2 \colon X \supset Z_1, \dots, X \supset Z_n$
- insert extra lines into Π_2 and use Modus Ponens, as follows:

1. if Z_i is axiom or member of S

   insert Z_i and $Z_i \supset (X \supset Z_i)$ before $X \supset Z_i$

2. if $Z_i = X$

   insert steps of proof of $X \supset Z_i$ before it

3. if Z_i is derived with Modus Ponens from Z_j and Z_k with $j, k < i$
   then $Z_k = (Z_j \supset Z_i)$
   insert $(X \supset (Z_j \supset Z_i)) \supset ((X \supset Z_j) \supset (X \supset Z_i))$ and
   $(X \supset Z_j) \supset (X \supset Z_i)$ before $X \supset Z_i$

- resulting sequence is derivation of $X \supset Y$ from S

Definition (Axiom Schemes 3 – 9)

3	$\bot \supset X$	7	$\alpha \supset \alpha_1$
4	$X \supset \top$	8	$\alpha \supset \alpha_2$
5	$\neg\neg X \supset X$	9	$(\beta_1 \supset X) \supset ((\beta_2 \supset X) \supset (\beta \supset X))$
6	$X \supset (\neg X \supset Y)$		

Example

$(\neg X \supset X) \supset X$ is theorem:

1.	$(\neg\neg X \supset X) \supset ((X \supset X) \supset ((\neg X \supset X) \supset X))$	Axiom Scheme 9
2.	$\neg\neg X \supset X$	Axiom Scheme 5
3.	$(X \supset X) \supset ((\neg X \supset X) \supset X)$	Modus Ponens
4.	$X \supset X$	earlier proof
5.	$(\neg X \supset X) \supset X$	Modus Ponens

Theorem (Strong Hilbert Soundness)

if $S \vdash_{ph} X$ *then* $S \vDash_p X$

Proof

- let $Z_1, \dots, Z_n$ be derivation of X from S, so $Z_n = X$
- we show $S \vDash_p Z_i$ by induction on i
  1. if Z_i is axiom then Z_i is tautology and thus also $S \vDash_p Z_i$
  2. if $Z_i \in S$ then $S \vDash_p Z_i$ holds trivially
  3. if Z_i is obtained from Z_j and Z_k by Modus Ponens then $Z_k = (Z_j \supset Z_i)$ and $j, k < i$

     $S \vDash_p Z_j$ and $S \vDash_p Z_j \supset Z_i$ follow from induction hypothesis

     $S \vDash_p Z_i$ follows from definition of $\vDash_p$

Definition

- set S of formulas is X–Hilbert inconsistent if $S \vdash_{ph} X$
- set S of formulas is X–Hilbert consistent if $S \vdash_{ph} X$ does not hold

Lemma

collection of all X–Hilbert consistent sets is propositional consistency property

Proof

let S be X–Hilbert consistent

1. if $A \in S$ and $\neg A \in S$ then $S \vdash_{ph} A$ and $S \vdash_{ph} \neg A$

   Axiom Scheme 6: $\vdash_{ph} A \supset (\neg A \supset X)$

   $S \vdash_{ph} X$ by two applications of Modus Ponens ↯

Proof (cont'd)

let S be X – Hilbert consistent

2. if $\bot \in S$ then $S \vdash_{ph} \bot$

   Axiom Scheme 3: $\vdash_{ph} \bot \supset X$

   $S \vdash_{ph} X$ by Modus Ponens ↯

   if $\neg\top \in S$ then $S \vdash_{ph} \neg\top$

   Axiom Scheme 4: $\vdash_{ph} \neg\top \supset \top$

   $S \vdash_{ph} \top$ by Modus Ponens

   Axiom Scheme 6: $\vdash_{ph} \top \supset (\neg\top \supset X)$

   $S \vdash_{ph} X$ by two applications of Modus Ponens ↯

Proof (cont'd)

let S be X–Hilbert consistent

3. if $\neg\neg Z \in S$ then $S \vdash_{ph} \neg\neg Z$

   Axiom Scheme 5: $\vdash_{ph} \neg\neg Z \supset Z$

   $S \vdash_{ph} Z$ by Modus Ponens

   if $S \cup \{Z\} \vdash_{ph} X$ then $S \vdash_{ph} Z \supset X$ by Deduction Theorem

   $S \vdash_{ph} X$ by Modus Ponens ↯

   $S \cup \{Z\}$ is X–Hilbert consistent

4. if $\alpha \in S$ then ... exercise ...

Proof (cont'd)

let S be $X-$Hilbert consistent

5 if $\beta \in S$ then $S \vdash_{ph} \beta$

suppose both $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ are $X-$Hilbert inconsistent

$S \cup \{\beta_1\} \vdash_{ph} X$ and $S \cup \{\beta_2\} \vdash_{ph} X$

$S \vdash_{ph} \beta_1 \supset X$ and $S \vdash_{ph} \beta_2 \supset X$ by Deduction Theorem

Axiom Scheme 9: $\vdash_{ph} (\beta_1 \supset X) \supset ((\beta_2 \supset X) \supset (\beta \supset X))$

$S \vdash_{ph} \beta \supset X$ by two applications of Modus Ponens

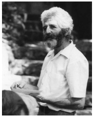
$S \vdash_{ph} X$ by Modus Ponens ↯

Theorem (Strong Hilbert Completeness)

if $S \models_p X$ *then* $S \vdash_{ph} X$

Proof

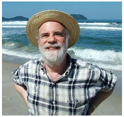
- suppose $S \vdash_{ph} X$ does not hold, so S is $X-$Hilbert consistent
- $\vdash_{ph} (\neg X \supset X) \supset X$
- if $S \cup \{\neg X\} \vdash_{ph} X$ then $S \vdash_{ph} \neg X \supset X$ and thus $S \vdash_{ph} X$ $\lightning$
- $S \cup \{\neg X\} \vdash_{ph} X$ does not hold
- $S \cup \{\neg X\}$ is $X-$Hilbert consistent
- $S \cup \{\neg X\}$ is satisfiable (by previous lemma and Model Existence Theorem)
- $S \models_p X$ does not hold



William Craig
(1918 – 2016)

David Hilbert
(1862 – 1943)

Jaakko Hintikka
(1929 – 2015)



Outline

Fitting

- Bonus: Exercise 3.6.6 or Exercise 3.6.7
  (where ‘or’ means that you can get at most 1 bonus-exercise point)
- Exercise 3.7.1
- Exercise 3.7.2.(1) and (2)
- Bonus: Exercise 3.7.4 (hence 3.7.3 and 3.7.2 as well)
- Exercise 3.9.1
- Bonus: Exercise 3.9.2 or Exercise 3.9.3
- Exercise 4.1.1
- Exercise 4.1.2 !
- Bonus: Exercise 4.1.4 or 4.1.5 or Exercise 4.1.6
- Exercise 4.1.7 !
- Exercise 4.1.8
- Bonus: Exercise 4.5.2

Outline

- Summary of Previous Lecture
- Model Existence Theorem
- Compactness
- Interpolation
- Semantic Tableaux
- Hilbert Systems
- Exercises
- Further Reading

Fitting

- Section 3.7 (until Theorem 3.7.3)
- Section 3.8 (until Corollary 3.8.2) !
- Section 3.9
- Section 4.1 !
- Section 4.5