Computational Logic

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Outline

- Overviews
- First-Order Logic
- Herbrand Models
- Uniform Notation
- Hintikka's Lemma
- Exercises
- Further Reading

Overview of previous lecture

The previous lecture was concerned with two things, namely (meta-theoretic) consequences of Model Existence and the presentation of Hilbert systems.

- The rough idea of Model Existence is based on maximally deriving consequences of a given set formulas, without arriving at a contradiction. Maximality is captured by the notion of a Hintikka set, which one can think of as expressing that the set is closed under taking consequences, i.e. closed under the tableau expansion rules, and is consistent in that it does not contain a formula and its negation, no contradiction. Hintikka's lemma expresses that such sets are satisfiable (think of satisying a branch of a tableau where we have exhousted all possible consequence without it being closed), and Model Existence that every set in a Propositional Consistency Property, a collection of sets of formulas having certain closure properties, is satisfiable because it can be extended to a Hintikka set (in the collection). From this completeness of tableaux follows, since if never a closed tableau is obtained (from the negation of a formula), a Propositional Consistency Property is obtained, so the negated formula is satisfiable, hence the original formula a tautology.

Overview of previous lecture (ctd)

- Compactness will be commented on below, for first-order logic.
- Interpolation is a meta-theoretic result having uses in model checking and database theory. One can think of it as expressing a restriction on the converse of transitivity: Whereas, transitivity expresses that given X implies Y and Y implies Z, it follows that X implies Z, Craig's interpolation theorem expresses that given X implies Z, one can find an interpolant Y such that X implies Y and Y implies Z, and that for finding Y we may restrict ourselves to formulas speaking about the variables X and Z; i.e. an interpolant/way-point Y can always be found using only variables that are in the common language of X and Z. Versions of interpolation hold e.g. for first-order logic (we will see) and equational logic. There are also logics for which interpolation does not hold, e.g. rewriting logic.

Overview of previous lecture (ctd)

- There is usually an exchange in inference systems, between having many axioms and few inference rules, and having few axioms and many inference rules. Hilbert systems are the extreme case of the former, just the inference rule of modus ponens, whereas natural deduction is an instance of the latter, having introduction and elimination rules for each connective. The deduction theorem connects both by showing that Y can be inferred from X iff there is a Hilbert proof of $X \supset Y$. The proof is constructive, so that to prove the latter, one can first prove the former, and then transform that into a Hilbert proof using the deduction theorem. The Hilbert system presented has only two axioms, but was shown complete using Model Existence, showing the versatility of the latter result.

Overview of this lecture

This lecture is concerned with generalising our set-up for to first-order logic, the default logic in automated reasoning, by allowing to express properties of and relations between individuals (predicates instead of just properties), and all ($\forall$) or some ($\exists$) of these. The items below, on this page, should be known.

- syntax: terms to represent individuals (e.g. natural numbers) and operations on them (e.g. addition), predicates to describe properties of individuals (unary predicates) and relations between them (binary, ternary, . . . predicates), and quantifiers to express properties holding for some/all individuals.
- semantics: meaning of terms is given by means of a domain from which individuals are taken and over which quantifiers range; operations are interpreted as functions on the domain, and predicates as relation on it. The meaning of a concrete formula depends on an assignment giving meaning to the variables in a term, as elements of the domain. This dependence is there to allow interpreting formulas having quantifiers by recursion on the formula. E.g. $(\forall x)\Phi$ is true if for all assignments to x, the subformula Φ is true. The semantics is then used to define the generalised notions of validity and satisfiability.

Overview of this lecture (ctd)

- Herbrand models are models where terms are interpreted as themselves. That is, Herbrand models are a kind of syntactic models; instead of taking as domain say the natural numbers, or people or . . . we take the terms themselves as individuals. For the Herbrand model, assignments are substitutions for the variables. Herbrand models are counterintuitive (aren't interpretations meant to give meaning/semantics to the language/syntax? how can syntax fulfill the role of semantics?) but (cf. free groups) they work: Herbrand models are typically used for meta-theoretic results connecting semantics to syntax, with the reasoning going roughly as follows: if a formula is valid, then it is true in all models, so in particular in the Herbrand model; thus (Herbrand's theorm), if the formula is existential it suffices to find appropriate termsfor the existentially quantified variables. This enables automation. Proof search by enumerating terms is the basis for Prolog (how would one check all interpretations in all domains?).
- Uniform notation is generalised by noting that $\forall$ is a generalised conjunction and $\exists$ a generalised disjunction. For example, one can think of $(\exists n)n \geq 5$ in the natural numbers as $(0 \geq 5) \vee (1 \geq 5) \vee (2 \geq 5) \ldots$.

Overview of this lecture (ctd)

- The meta-theoretic results follow by generalising the propositional case, or rather the cpnverse: Hintikka and Model Existence were set-up for the propositional case such that they would allow easy generalisation to the first-order case. In the propositional case one could often do with finite sets of propositions (e.g. for showing completeness results), but that would not do for the first-order case. An intuition for this insufficiency is provided by the above correspondence between quantified formulas and infinite con/disjunctions.
- Compactness is a meta-theoretic result in that it can be used to show the limitations of first-order logic. In particular, it implies that there is no first-order formula that can express that the domain is finite. From the Löwenheim–Skolem theorem follows a similar limitation of first-order logic: whatever first-order axiomatisation one gives of the real numbers, there will always be a countable model. This means that first-order formulas cannot capture the uncountability of the real numbers.

Part I: Propositional Logic

compactness, completeness, Hilbert systems, Hintikka's lemma, interpolation, logical consequence, model existence theorem, propositional semantic tableaux, soundness

Part II: First-Order Logic

compactness, completeness, Craig's interpolation theorem, cut elimination, first-order semantic tableaux, Herbrand models, Herbrand's theorem, Hilbert systems, Hintikka's lemma, Löwenheim-Skolem, logical consequence, model existence theorem, prenex form, skolemization, soundness

Part III: Limitations and Extensions of First-Order Logic

Curry-Howard isomorphism, intuitionistic logic, Kripke models, second-order logic, simply-typed λ-calculus

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notation: $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ (or simply L)

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4. if A is formula of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ and x is variable then $(\forall x)A$ and $(\exists x)A$ are formulas of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$

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variable occurrence is **bound** if it is not free

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$(\forall x)[(\exists y)R(f(x,y),c) \supset (\exists z)S(y,z)]$

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sentence (or closed formula) of $L(\mathbf{R},\mathbf{F},\mathbf{C})$ is formula of $L(\mathbf{R},\mathbf{F},\mathbf{C})$ with no free-variable occurrences

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composition of substitutions σ and τ is substitution $\sigma\tau$ such that $x(\sigma\tau) = (x\sigma)\tau$ for each variable x

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composition of substitutions having finite support is substitution having finite support

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2. $[\neg X]\sigma = \neg[X\sigma]$

Notation

$\{x_1/t_1, \ldots, x_n/t_n\}$ for substitution σ having finite support $\{x_1, \ldots, x_n\}$ and $x_i\sigma = t_i$ for $1 \leqslant i \leqslant n$

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given substitution σ and variable x, substitution σ_x is defined as follows:

$y\sigma_x = x$ if $y = x$ and $y\sigma_x = y\sigma$ if $y \neq x$

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Example

if $\sigma = \{x/a, y/b\}$ then

$$[(\forall x)R(x,y) \supset (\exists y)R(x,y)]\sigma$$

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$$[(\forall x)R(x,y) \supset (\exists y)R(x,y)]\sigma \;=\; [(\forall x)R(x,y)]\sigma \supset [(\exists y)R(x,y)]\sigma$$

Example

if $\sigma = \{x/a, y/b\}$ then

$$\begin{aligned}[(\forall x)R(x,y) \supset (\exists y)R(x,y)]\sigma &= [(\forall x)R(x,y)]\sigma \supset [(\exists y)R(x,y)]\sigma \\ &= (\forall x)[R(x,y)]\sigma_x \supset (\exists y)[R(x,y)]\sigma_y\end{aligned}$$

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Remark

$(\Phi\sigma)\tau = \Phi(\sigma\tau)$ need not hold

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if $\sigma = \{x/a, y/b\}$ then

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- $\sigma\tau = \{x/c, y/c\}$

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if $\sigma = \{x/a, y/b\}$ then

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Remark

$(\Phi\sigma)\tau = \Phi(\sigma\tau)$ need not hold

Example

$\Phi = (\forall y)R(x,y)$, $\sigma = \{x/y\}$, $\tau = \{y/c\}$

- $(\Phi\sigma)\tau = (\forall y)R(y,y)\tau = (\forall y)R(y,y)$
- $\sigma\tau = \{x/c, y/c\}$
- $\Phi(\sigma\tau) = (\forall y)R(c,y)$

Definition

substitution σ being free for formula is defined as follows:

1. if A is atomic then σ free for A

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   - if y is free variable of Φ other than x, $y\sigma$ does not contain x

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   - σ_x is free for Φ
   - if y is free variable of Φ other than x, $y\sigma$ does not contain x

Theorem

if substitution σ is free for formula X and substitution τ is free for $X\sigma$ then $(X\sigma)\tau = X(\sigma\tau)$

Proof

structural induction on X

- atomic case is obvious

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- $X = \neg Y$ $\quad \sigma$ is free for Y

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structural induction on X

- atomic case is obvious
- $X = \neg Y$ $\quad \sigma$ is free for Y $\quad X\sigma = \neg(Y\sigma)$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y$ σ is free for Y $X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y \quad \sigma$ is free for $Y \quad X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y$ σ is free for Y $X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y$ $\quad$ σ is free for Y $\quad$ $X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau)$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y \quad \sigma$ is free for $Y \quad X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau))$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y$ σ is free for Y $X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau)$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y$ $\quad \sigma$ is free for Y $\quad X\sigma = \neg(Y\sigma)$

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  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau) = X(\sigma\tau)$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y$ $\quad \sigma$ is free for Y $\quad X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau) = X(\sigma\tau)$
- $X = (Y \circ Z)$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y$ $\quad \sigma$ is free for Y $\quad X\sigma = \neg(Y\sigma)$

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  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau) = X(\sigma\tau)$
- $X = (Y \circ Z)$ $\quad \sigma$ is free for Y and Z

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y \quad \sigma$ is free for $Y \quad X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau) = X(\sigma\tau)$
- $X = (Y \circ Z) \quad \sigma$ is free for Y and $Z \quad X\sigma = (Y\sigma \circ Z\sigma)$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y$ $\quad \sigma$ is free for Y $\quad X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau) = X(\sigma\tau)$
- $X = (Y \circ Z)$ $\quad \sigma$ is free for Y and Z $\quad X\sigma = (Y\sigma \circ Z\sigma)$

  τ is free for $Y\sigma$ and $Z\sigma$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y \quad \sigma$ is free for $Y \quad X\sigma = \neg(Y\sigma)$

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  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau) = X(\sigma\tau)$
- $X = (Y \circ Z) \quad \sigma$ is free for Y and $Z \quad X\sigma = (Y\sigma \circ Z\sigma)$

  τ is free for $Y\sigma$ and $Z\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ and $(Z\sigma)\tau = Z(\sigma\tau)$ follow from induction hypothesis

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  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau) = X(\sigma\tau)$
- $X = (Y \circ Z)$ $\quad \sigma$ is free for Y and Z $\quad X\sigma = (Y\sigma \circ Z\sigma)$

  τ is free for $Y\sigma$ and $Z\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ and $(Z\sigma)\tau = Z(\sigma\tau)$ follow from induction hypothesis

  $(X\sigma)\tau = ((Y\sigma)\tau \circ (Z\sigma)\tau)$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y$ σ is free for Y $X\sigma = \neg(Y\sigma)$

  τ is free for $Y\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau) = X(\sigma\tau)$
- $X = (Y \circ Z)$ σ is free for Y and Z $X\sigma = (Y\sigma \circ Z\sigma)$

  τ is free for $Y\sigma$ and $Z\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ and $(Z\sigma)\tau = Z(\sigma\tau)$ follow from induction hypothesis

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  $(X\sigma)\tau = [\neg(Y\sigma)]\tau = \neg((Y\sigma)\tau) = \neg(Y(\sigma\tau)) = (\neg Y)(\sigma\tau) = X(\sigma\tau)$
- $X = (Y \circ Z) \quad \sigma$ is free for Y and $Z \quad X\sigma = (Y\sigma \circ Z\sigma)$

  τ is free for $Y\sigma$ and $Z\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ and $(Z\sigma)\tau = Z(\sigma\tau)$ follow from induction hypothesis

  $(X\sigma)\tau = ((Y\sigma)\tau \circ (Z\sigma)\tau) = (Y(\sigma\tau) \circ Z(\sigma\tau)) = (Y \circ Z)(\sigma\tau)$

Proof

structural induction on X

- atomic case is obvious
- $X = \neg Y \quad \sigma$ is free for $Y \quad X\sigma = \neg(Y\sigma)$

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  $(Y\sigma)\tau = Y(\sigma\tau)$ follows from induction hypothesis

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- $X = (Y \circ Z) \quad \sigma$ is free for Y and $Z \quad X\sigma = (Y\sigma \circ Z\sigma)$

  τ is free for $Y\sigma$ and $Z\sigma$

  $(Y\sigma)\tau = Y(\sigma\tau)$ and $(Z\sigma)\tau = Z(\sigma\tau)$ follow from induction hypothesis

  $(X\sigma)\tau = ((Y\sigma)\tau \circ (Z\sigma)\tau) = (Y(\sigma\tau) \circ Z(\sigma\tau)) = (Y \circ Z)(\sigma\tau) = X(\sigma\tau)$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi \quad \sigma_x$ is free for Φ

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi \quad \sigma_x$ is free for $\Phi \quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi$ $\quad \sigma_x$ is free for Φ $\quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

  τ_x is free for $\Phi\sigma_x$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi \quad \sigma_x$ is free for $\Phi \quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

  τ_x is free for $\Phi\sigma_x$

  $(\Phi\sigma_x)\tau_x = \Phi(\sigma_x\tau_x)$ follows from induction hypothesis

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi$ $\quad \sigma_x$ is free for Φ $\quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

  τ_x is free for $\Phi\sigma_x$

  $(\Phi\sigma_x)\tau_x = \Phi(\sigma_x\tau_x)$ follows from induction hypothesis

  claim: $\Phi(\sigma_x\tau_x) = \Phi(\sigma\tau)_x$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi \quad \sigma_x$ is free for $\Phi \quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

  τ_x is free for $\Phi\sigma_x$

  $(\Phi\sigma_x)\tau_x = \Phi(\sigma_x\tau_x)$ follows from induction hypothesis

  claim: $\Phi(\sigma_x\tau_x) = \Phi(\sigma\tau)_x$

  let y be free variable of Φ

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi$ $\quad \sigma_x$ is free for Φ $\quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

  τ_x is free for $\Phi\sigma_x$

  $(\Phi\sigma_x)\tau_x = \Phi(\sigma_x\tau_x)$ follows from induction hypothesis

  claim: $\Phi(\sigma_x\tau_x) = \Phi(\sigma\tau)_x$

  let y be free variable of Φ

  - if $y = x$
  - if $y \neq x$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi$ $\quad \sigma_x$ is free for Φ $\quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

  τ_x is free for $\Phi\sigma_x$

  $(\Phi\sigma_x)\tau_x = \Phi(\sigma_x\tau_x)$ follows from induction hypothesis

  claim: $\Phi(\sigma_x\tau_x) = \Phi(\sigma\tau)_x$

  let y be free variable of Φ

  - if $y = x$ then $y(\sigma_x\tau_x) = (y\sigma_x)\tau_x$
  - if $y \neq x$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi \quad \sigma_x$ is free for $\Phi \quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

  τ_x is free for $\Phi\sigma_x$

  $(\Phi\sigma_x)\tau_x = \Phi(\sigma_x\tau_x)$ follows from induction hypothesis

  claim: $\Phi(\sigma_x\tau_x) = \Phi(\sigma\tau)_x$

  let y be free variable of Φ

  - if $y = x$ then $y(\sigma_x\tau_x) = (y\sigma_x)\tau_x = x\tau_x$
  - if $y \neq x$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi$ $\quad \sigma_x$ is free for Φ $\quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

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  let y be free variable of Φ

  - if $y = x$ then $y(\sigma_x\tau_x) = (y\sigma_x)\tau_x = x\tau_x = x$
  - if $y \neq x$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi$ $\quad \sigma_x$ is free for Φ $\quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

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  $(\Phi\sigma_x)\tau_x = \Phi(\sigma_x\tau_x)$ follows from induction hypothesis

  claim: $\Phi(\sigma_x\tau_x) = \Phi(\sigma\tau)_x$

  let y be free variable of Φ

  - if $y = x$ then $y(\sigma_x\tau_x) = (y\sigma_x)\tau_x = x\tau_x = x = y(\sigma\tau)_x$
  - if $y \neq x$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi \quad \sigma_x$ is free for $\Phi \quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

  τ_x is free for $\Phi\sigma_x$

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  - if $y = x$ then $y(\sigma_x\tau_x) = (y\sigma_x)\tau_x = x\tau_x = x = y(\sigma\tau)_x$
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Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi$ $\quad \sigma_x$ is free for Φ $\quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

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  - if $y = x$ then $y(\sigma_x\tau_x) = (y\sigma_x)\tau_x = x\tau_x = x = y(\sigma\tau)_x$
  - if $y \neq x$ then $y(\sigma_x\tau_x) = (y\sigma_x)\tau_x = (y\sigma)\tau_x$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi \quad \sigma_x$ is free for $\Phi \quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

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  let y be free variable of Φ

  - if $y = x$ then $y(\sigma_x\tau_x) = (y\sigma_x)\tau_x = x\tau_x = x = y(\sigma\tau)_x$
  - if $y \neq x$ then $y(\sigma_x\tau_x) = (y\sigma_x)\tau_x = (y\sigma)\tau_x = (y\sigma)\tau$

Proof (cont'd)

structural induction on X

- $X = (\forall x)\Phi$ $\quad \sigma_x$ is free for Φ $\quad \tau$ is free for $[(\forall x)\Phi]\sigma = (\forall x)[\Phi\sigma_x]$

  τ_x is free for $\Phi\sigma_x$

  $(\Phi\sigma_x)\tau_x = \Phi(\sigma_x\tau_x)$ follows from induction hypothesis

  claim: $\Phi(\sigma_x\tau_x) = \Phi(\sigma\tau)_x$

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$$(X\sigma)\tau = ((\forall x)[\Phi\sigma_x])\tau$$

Proof (cont'd)

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$$(X\sigma)\tau = ((\forall x)[\Phi\sigma_x])\tau = (\forall x)[(\Phi\sigma_x)\tau_x]$$

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Proof (cont'd)

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- $X = (\exists x)\Phi$ $\quad$ similar

Outline

Definition

model for first-order language $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is pair $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$

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  - to every n-place $f \in \mathbf{F}$ some n-ary function $f^{\mathbf{I}} \colon \mathbf{D}^n \to \mathbf{D}$

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assignment in model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ is mapping $\mathbf{A}$ from set of variables to set $\mathbf{D}$

image of variable v under assignment $\mathbf{A}$ is denoted by $v^{\mathbf{A}}$

Definition

given model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ for language $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ and assignment $\mathbf{A}$ in $\mathbf{M}$, value $t^{\mathbf{I},\mathbf{A}}$ in $\mathbf{D}$ is defined inductively:

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3. $[f(t_1, \dots, t_n)]^{\mathbf{I},\mathbf{A}} = f^{\mathbf{I}}(t_1^{\mathbf{I},\mathbf{A}}, \dots, t_n^{\mathbf{I},\mathbf{A}})$ for n-place function symbol f

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Example

L with constant 0, one-place function symbol s, two-place function symbol $+$

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Example

L with constant 0, one-place function symbol s, two-place function symbol $+$

terms $t_1 = s(s(0) + s(x))$ and $t_2 = s(x + s(x + s(0)))$

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model $\mathbf{M}_1 = \langle \mathbf{D}, \mathbf{I} \rangle$ with $\mathbf{D} = \mathbb{N}$, $0^{\mathbf{I}} = 0$, $s^{\mathbf{I}}(n) = n + 1$, $+^{\mathbf{I}}(m, n) = m + n$

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assignment $\mathbf{A}$ with $x^{\mathbf{A}} = 3$

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L with constant 0, one-place function symbol s, two-place function symbol $+$

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$t_1^{\mathbf{I},\mathbf{A}} = 6$

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assignment $\mathbf{A}$ with $x^{\mathbf{A}} = 3$

$t_1^{\mathbf{I},\mathbf{A}} = 6$ and $t_2^{\mathbf{I},\mathbf{A}} =$

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assignment $\mathbf{A}$ with $x^{\mathbf{A}} = 3$

$t_1^{\mathbf{I},\mathbf{A}} = 6$ and $t_2^{\mathbf{I},\mathbf{A}} = 9$

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model $\mathbf{M}_2 = \langle \mathbf{D}, \mathbf{I} \rangle$ with $\mathbf{D} = \{a, b\}^*$, $0^{\mathbf{I}} = a$, $s^{\mathbf{I}}(w) = wa$, $+^{\mathbf{I}}(v, w) = vw$

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L with constant 0, one-place function symbol s, two-place function symbol $+$

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assignment $\mathbf{A}$ with $x^{\mathbf{A}} = aba$

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Example

L with constant 0, one-place function symbol s, two-place function symbol $+$

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assignment $\mathbf{A}$ with $x^{\mathbf{A}} = aba$

$t_1^{\mathbf{I},\mathbf{A}} =$ and $t_2^{\mathbf{I},\mathbf{A}} =$

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assignment $\mathbf{A}$ with $x^{\mathbf{A}} = aba$

$t_1^{\mathbf{I},\mathbf{A}} = aaabaaa$ and $t_2^{\mathbf{I},\mathbf{A}} =$

Definition

given model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ for language $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ and assignment $\mathbf{A}$ in $\mathbf{M}$, value $t^{\mathbf{I},\mathbf{A}}$ in $\mathbf{D}$ is defined inductively:

1. $c^{\mathbf{I},\mathbf{A}} = c^{\mathbf{I}}$ for constant symbol c
2. $v^{\mathbf{I},\mathbf{A}} = v^{\mathbf{A}}$ for variable v
3. $[f(t_1, \dots, t_n)]^{\mathbf{I},\mathbf{A}} = f^{\mathbf{I}}(t_1^{\mathbf{I},\mathbf{A}}, \dots, t_n^{\mathbf{I},\mathbf{A}})$ for n-place function symbol f

Example

L with constant 0, one-place function symbol s, two-place function symbol $+$

terms $t_1 = s(s(0) + s(x))$ and $t_2 = s(x + s(x + s(0)))$

model $\mathbf{M}_2 = \langle \mathbf{D}, \mathbf{I} \rangle$ with $\mathbf{D} = \{a, b\}^*$, $0^{\mathbf{I}} = a$, $s^{\mathbf{I}}(w) = wa$, $+^{\mathbf{I}}(v, w) = vw$

assignment $\mathbf{A}$ with $x^{\mathbf{A}} = aba$

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1. $[P(t_1, \dots, t_n)]^{\mathbf{I},\mathbf{A}} = \mathrm{t} \iff \langle t_1^{\mathbf{I},\mathbf{A}}, \dots, t_n^{\mathbf{I},\mathbf{A}} \rangle \in P^{\mathbf{I}}$

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Notation

$\Phi^{\mathbf{I}}$ instead of $\Phi^{\mathbf{I},\mathbf{A}}$ for formulas Φ without free variables

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Example

L with two-place function symbol $\oplus$ and two-place relation symbol R

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Lemma

given closed term t, formula Φ of first-order language L, model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ for L

if x is variable and $\mathbf{A}$ assignment such that $x^{\mathbf{A}} = t^{\mathbf{I}}$ then $[\Phi\{x/t\}]^{\mathbf{I},\mathbf{B}} = \Phi^{\mathbf{I},\mathbf{A}}$ for any x-variant $\mathbf{B}$ of $\mathbf{A}$

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given model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ *for language* L*, formula* Φ *in* L*, assignment* $\mathbf{A}$ *in* $\mathbf{M}$*, substitution* σ *that is free for* Φ

if assignment $\mathbf{B}$ *is defined by* $v^{\mathbf{B}} = (v\sigma)^{\mathbf{I},\mathbf{A}}$ *for each variable* v *then* $\Phi^{\mathbf{I},\mathbf{B}} = (\Phi\sigma)^{\mathbf{I},\mathbf{A}}$

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Outline

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. . . exercise . . .

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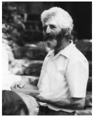
... exercise ...



William Craig
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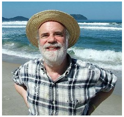

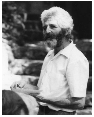

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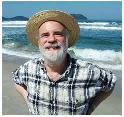

Outline

Definition

universal		existential	
γ		δ	
$(\forall x)\Phi$		$(\exists x)\Phi$	
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set S *of sentences, sentences* γ *and* δ

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1. suppose $S \cup \{\gamma\}$ is satisfiable in model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$

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   let $\mathbf{A}$ be any assignment such that $x^{\mathbf{A}} = t^{\mathbf{I}}$

   $[\gamma(t)]^{\mathbf{I},\mathbf{A}} = [\gamma\{x/t\}]^{\mathbf{I},\mathbf{A}} = [\gamma(x)]^{\mathbf{I},\mathbf{A}} = \mathrm{t}$ (using Lemma on slide 43)

2. suppose $S \cup \{\delta\}$ is satisfiable in model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$

   $(\exists x)\delta(x)$ is true in $\mathbf{M}$ (with x new to δ)

   $[\delta(x)]^{\mathbf{I},\mathbf{A}}$ is true for some assignment $\mathbf{A}$

   construct new model $\mathbf{M}^* = \langle \mathbf{D}, \mathbf{J} \rangle$ with $\mathbf{J}$ identical to $\mathbf{I}$ except $p^{\mathbf{J}} = x^{\mathbf{A}}$

   $S \cup \{\delta\}$ is satisfiable in $\mathbf{M}^*$ and $[\delta(x)]^{\mathbf{J},\mathbf{A}}$ is true

   $[\delta(p)]^{\mathbf{J},\mathbf{A}} = [\delta\{x/p\}]^{\mathbf{J},\mathbf{A}}$

Proof

1. suppose $S \cup \{\gamma\}$ is satisfiable in model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$

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   let $\mathbf{A}$ be any assignment such that $x^{\mathbf{A}} = t^{\mathbf{I}}$

   $[\gamma(t)]^{\mathbf{I},\mathbf{A}} = [\gamma\{x/t\}]^{\mathbf{I},\mathbf{A}} = [\gamma(x)]^{\mathbf{I},\mathbf{A}} = \mathrm{t}$ (using Lemma on slide 43)

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   $[\delta(x)]^{\mathbf{I},\mathbf{A}}$ is true for some assignment $\mathbf{A}$

   construct new model $\mathbf{M}^* = \langle \mathbf{D}, \mathbf{J} \rangle$ with $\mathbf{J}$ identical to $\mathbf{I}$ except $p^{\mathbf{J}} = x^{\mathbf{A}}$

   $S \cup \{\delta\}$ is satisfiable in $\mathbf{M}^*$ and $[\delta(x)]^{\mathbf{J},\mathbf{A}}$ is true

   $[\delta(p)]^{\mathbf{J},\mathbf{A}} = [\delta\{x/p\}]^{\mathbf{J},\mathbf{A}} = [\delta(x)]^{\mathbf{J},\mathbf{A}} = \mathrm{t}$ (using Lemma on slide 43)

   $S \cup \{\delta, \delta(p)\}$ is satisfiable (in $\mathbf{M}^*$)

Definition

rank $r(X)$ of first-order formula: $r(A) = r(\neg A) = r(\top) = r(\bot) = 0$

$$r(\neg\top) = r(\neg\bot) = 1 \qquad r(\neg\neg Z) = r(Z) + 1 \qquad r(\alpha) = r(\alpha_1) + r(\alpha_2) + 1$$

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Theorem (First-Order Structural Induction)

every formula of first-order language L has property Q provided

- *basis step*

  every atomic formula and its negation has property Q

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Lemma

if $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$ *is Herbrand model for* L *then*

- *formula* γ *of* L *is true in* $\mathbf{M} \iff \gamma(d)$ *is true in* $\mathbf{M}$ *for every* $d \in \mathbf{D}$

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Proof

. . . exercise . . .

Outline

Definition

set **H** of sentences of first-order language L is first-order Hintikka set, provided

1. for any propositional letter A, not both $A \in \mathbf{H}$ and $\neg A \in \mathbf{H}$
2. $\bot \notin \mathbf{H}$, $\neg\top \notin \mathbf{H}$
3. if $\neg\neg Z \in \mathbf{H}$ then $Z \in \mathbf{H}$
4. if $\alpha \in \mathbf{H}$ then $\alpha_1 \in \mathbf{H}$ and $\alpha_2 \in \mathbf{H}$
5. if $\beta \in \mathbf{H}$ then $\beta_1 \in \mathbf{H}$ or $\beta_2 \in \mathbf{H}$

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Lemma (Hintikka's Lemma)

if $\mathbf{H}$ *is first-order Hintikka set with respect to language* L *with nonempty set of closed terms then* $\mathbf{H}$ *is satisfiable in Herbrand model*

Proof

- **H** is first-order Hintikka set with respect to L

Proof

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- construct model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$

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- **H** is first-order Hintikka set with respect to L
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Proof

- $\mathbf{H}$ is first-order Hintikka set with respect to L
- construct model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$
  - $\mathbf{D}$ is collection of closed terms of L
  - $c^{\mathbf{I}} = c$ for constant symbols c of L

Proof

- $\mathbf{H}$ is first-order Hintikka set with respect to L
- construct model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$
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  - $c^{\mathbf{I}} = c$ for constant symbols c of L
  - $f^{\mathbf{I}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$ for n-place function symbols f of L and $t_1, \dots, t_n \in D$

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- construct model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$
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  - $\langle t_1, \dots, t_n \rangle$ belongs to $R^{\mathbf{I}}$ for n-place relation symbols R of L if $R(t_1, \dots, t_n) \in \mathbf{H}$
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- claim: if $X \in \mathbf{H}$ then X is true in $\mathbf{M}$, for every sentence X of L

Proof (cont'd)

claim: if $X \in \mathbf{H}$ then X is true in $\mathbf{M}$, for each sentence X of L

induction on X

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induction on X

- $\top$, $\bot$: trivial
- suppose $R(t_1, \dots, t_n) \in \mathbf{H}$

  $[R(t_1, \dots, t_n)]^{\mathbf{I},\mathbf{A}} = \mathrm{t}$ because $\langle t_1^{\mathbf{I},\mathbf{A}}, \dots, t_n^{\mathbf{I},\mathbf{A}} \rangle = \langle t_1, \dots, t_n \rangle \in R^{\mathbf{I}}$

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- α and β are treated as in propositional case

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- suppose $\gamma \in \mathbf{H}$

  $\gamma(t) \in \mathbf{H}$ for every closed term t

  $\gamma(t)$ is true in $\mathbf{M}$ for every $t \in \mathbf{D}$ according to induction hypothesis

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- suppose $R(t_1, \dots, t_n) \in \mathbf{H}$

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  γ is true in $\mathbf{M}$ using Lemma on slide 55

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- suppose $R(t_1, \dots, t_n) \in \mathbf{H}$

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  γ is true in $\mathbf{M}$ using Lemma on slide 55
- δ ... exercise ...

Outline

Definition

given first-order language $L = L(\mathbf{R}, \mathbf{F}, \mathbf{C})$

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collection $\mathcal{C}$ of sets of sentences of $L^{\mathbf{par}}$ is first-order consistency property if $\mathcal{C}$ is propositional consistency property and for each $S \in \mathcal{C}$:

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6. if $\gamma \in S$ then $S \cup \{\gamma(t)\} \in \mathcal{C}$ for every closed term t of $L^{\mathbf{par}}$
7. if $\delta \in S$ then $S \cup \{\delta(p)\} \in \mathcal{C}$ for some parameter p of $L^{\mathbf{par}}$

Theorem (First-Order Model Existence)

if $\mathcal{C}$ is first-order consistency property with respect to L and $S \in \mathcal{C}$ is set of sentences over L then S is satisfiable

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Lemma

every first-order consistency property can be extended to one that is subset closed

Theorem (First-Order Model Existence)

if $\mathcal{C}$ is first-order consistency property with respect to L and $S \in \mathcal{C}$ is set of sentences over L then S is satisfiable in Herbrand model with respect to $L^{\mathbf{par}}$

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Definition

alternate first-order consistency property is collection **C** meeting conditions for first-order consistency property, except that condition 7 is replaced by

7' if $\delta \in S$ then $S \cup \{\delta(p)\} \in \mathcal{C}$ for every parameter p that is new to S

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Lemma

every subset closed alternate first-order consistency property can be extended to one of finite character

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- $S \subseteq \mathbf{H}$ is satisfiable in Herbrand model with respect to $L^{\mathbf{par}}$

Outline

- Overviews
- First-Order Logic
- Herbrand Models
- Uniform Notation
- Hintikka's Lemma
  - Model Existence Theorem
  - Compactness
  - Löwenheim-Skolem
  - Logical Consequence
- Exercises
- Further Reading

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- suppose S is satisfiable in arbitrary large finite models
- let R be two-place relation symbol not in L and let L' be L extended with R
- there exist sentences $A_2, A_3, \ldots$ involving R such that A_i is not true in any model with less than i elements but can be made true in any domain with at least i elements

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Example

sentence

$$A_n \;=\; (\exists x_1)(\exists x_2)\cdots(\exists x_n)\Big[\;\bigwedge_{i=1}^{n} R(x_i,x_i) \;\wedge \bigwedge_{1\leqslant i<j\leqslant n} \neg R(x_i,x_j)\;\Big]$$

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Remark

notion of finiteness cannot be captured using machinery of classical first-order logic

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if set S of sentences of first-order language L is satisfiable then S is satisfiable in countable model

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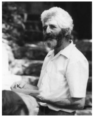
- $\mathcal{C}$ is first-order consistency property
- Model Existence Theorem:
  S is satisfiable in Herbrand model with respect to $L^{\textbf{par}}$
- $L^{\textbf{par}}$ has countable alphabet and hence countably many closed terms



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(1918 – 2016)

Jacques Herbrand
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David Hilbert
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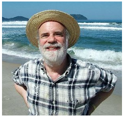

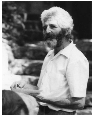

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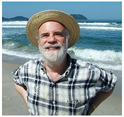

Leopold Löwenheim
(1878 – 1957)

Thoralf Skolem
(1887 – 1963)

Outline

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$\Leftarrow$ easy

Outline

Fitting

- Give a translation from propositional logic into first-order logic that is natural in the sense that properties carry over, e.g. that a formula is satisfiable/valid/a contradiction iff so is its translation.
- Exercise 5.2.2
- Bonus Exercise 5.2.4
- Exercise 5.3.2
- Exercise 5.3.9 !
- Exercise 5.4.1 or Exercise 5.4.2
- Bonus Exercise 5.5.2 or Exercise 5.6.3
- Exercise 5.9.2
- Bonus Exercise 5.9.3
- Exercise 5.10.1 or 5.10.3(1,2)

Outline

Fitting

- Section 5.7
- Section 5.8 !
- Section 5.9 !
- Section 5.10
- Section 6.1 !
- Section 6.3 !
- Section 6.4 !
- Section 6.5 !
- Section 8.2