Computational Logic

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Outline

- Overview of this Lecture
- First-Order Semantic Tableaux
- First-Order Hilbert Systems
- Replacement Theorem
- Skolemization
- Herbrand's Theorem
- Exercises
- Further Reading

Having set up the basic meta-theory for 1st-order logic, by generalising that for propositional logic, in particular Model Existence, we now focus on doing the same for their proof systems, tableaux and Hilbert Systems.

- The idea to generalize the tableau expansion rules from propositional to 1st-order logic, is that γ-formulas ($\forall$) generalise α-formulas (conjunction), and δ-formulas ($\exists$) generalise β-formulas (disjunction).

  Starting with the latter, since we need to keep our tableaux finite thinking of δ-formulas as infinite disjunctions will not do. What is done instead, is to have only one branch but with a new parameter for the variable bound by the $\exists$. The parameter being new guarantees that when closing that branch, entails the branch is closed for each of the infinitely many possible branches obtained by instantiating the bound variable, uniformly.

  Also for the latter we need to keep our tableaux finite, so thinking of γ-formulas as infinite conjunctions will not do either. What is done instead, is to judiciously choose an instance of the variable bound by the $\forall$. For choosing an instance, we use elements of the Herbrand model, i.e. closed terms. Although different choices for instantiating a γ-formula may be necessary along a branch, only finitely many such will be necessary.

- To generalise Hilbert Systems from propositional to 1st-order logic, we adjoin the Universal Generalization inference rule to deal with the $\forall$-quantifier, with the idea that if we can infer that a formula is a consequence for an arbitrary instance of the variable bound by the $\forall$, then also the $\forall$-formula is a consequence. Again, arbitrary instances are modelled by means of a sufficiently new parameter.

Next we consider the effect of syntactical transformations on the semantics.

- as for propositional logic, subformulas may be replaced by equivalent ones, without changing the meaning of the whole formula (replacement);
- occurrences of subformulas can be classified as being negative or positive, with the idea that if we make a positive occurrence of a subformula 'more true' then the formula as a whole becomes 'more true', whereas for negative subformulas it's the opposite (implicational replacement). An occurrence is positive if it is reached from the root by passing an even number of negations and negative otherwise; in $X \supset Y$, X occurs negative, Y positive. A subformula $(\forall x)\Phi$ occurring negatively 'is essentially an $\exists$' (it would be one after transformation into negation normal form). Similarly, a negative occurrence of an $\exists$ 'is essentially a $\forall$'.

- The idea of Skolemisation is to do away with existentially quantified variables, at the expense of introducing new function symbols. For instance, $(\forall x)(\exists y)(x < y)$ being true in, say, the natural numbers means there exists a function f such that $(\forall x)(x < f(x))$; we may take for f e.g. the $+1$ or $+17$ functions. Note that f must be a function having as arguments the variables that have been universally quantified 'before the $\exists$', in order to capture that the choice made to make the $\exists$ true may depend on these variables. An occurrence of a quantifier can be Skolemised, i.e. replaced by a function as sketched above, if it is an exists ($\exists$) and is positive, or a for all ($\forall$) and is negative (so 'essentially an $\exists$), preserving satisfiability.
- Whereas Skolemisation allows to get rid of (essentially) existential quantifiers, at the expense of introducing function symbols, Herbrand's theorem allows us to get rid of (essentially) universal quantifiers, at the expense of expanding them for a given (finite) set of closed terms, substituting each element of the set for the bound variable and taking the conjunction of all these choices. Roughly speaking, the combined effect of both is that we have gotten rid of quantifiers so can proceed 'as if we were in propositional logic'.

Part I: Propositional Logic

compactness, completeness, Hilbert systems, Hintikka's lemma, interpolation, logical consequence, model existence theorem, propositional semantic tableaux, soundness

Part II: First-Order Logic

compactness, completeness, Craig's interpolation theorem, cut elimination, first-order semantic tableaux, Herbrand models, Herbrand's theorem, Hilbert systems, Hintikka's lemma, Löwenheim-Skolem, logical consequence, model existence theorem, prenex form, skolemization, soundness

Part III: Limitations and Extensions of First-Order Logic

Curry-Howard isomorphism, intuitionistic logic, Kripke models, second-order logic, simply-typed λ-calculus

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Remark

first-order tableaux use sentences of $L^{\textbf{par}}$ to prove sentences of L

First-Order Tableau Expansion Rules

$$\frac{\neg\neg Z}{Z} \qquad \frac{\neg\bot}{\top} \qquad \frac{\neg\top}{\bot} \qquad \frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta}{\beta_1 \mid \beta_2} \qquad \frac{\gamma}{\gamma(t)} \qquad \frac{\delta}{\delta(p)}$$

for any closed term t of $L^{\textbf{par}}$ and new parameter p

Definitions

- S-introduction rule for tableaux: any member of S can be added to end of any tableau branch
- $S \vdash_{ft} X$ is there exists closed first-order tableau for $\{\neg X\}$, allowing S-introduction rule

Example

tableau proof of $(\forall x)[P(x) \vee Q(x)] \supset [(\exists x)P(x) \vee (\forall x)Q(x)]$:

$$\neg\big((\forall x)[P(x) \vee Q(x)] \supset [(\exists x)P(x) \vee (\forall x)Q(x)]\big)$$

$$(\forall x)[P(x) \vee Q(x)]$$

$$\neg\big[(\exists x)P(x) \vee (\forall x)Q(x)\big]$$

$$\neg(\exists x)P(x)$$

$$\neg(\forall x)Q(x)$$

$$\neg Q(c)$$

$$\neg P(c)$$

$$P(c) \vee Q(c)$$

$$P(c) \qquad\qquad Q(c)$$

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Definitions

- tableau branch θ is S-satisfiable if union of S and set of first-order sentences on θ is satisfiable
- tableau is S-satisfiable if some branch is S-satisfiable

Lemmata

- *any application of Tableau Expansion Rule as well as S-introduction rule to S-satisfiable tableau yields another S-satisfiable tableau*
- *there are no closed S-satisfiable tableaux*

Theorem (Strong Soundness)

if $S \vdash_{ft} X$ then $S \vDash_f X$

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Definition

finite set S of sentences of $L^{\mathbf{par}}$ is tableau consistent if S has no closed tableau

Lemma

collection of all tableau consistent sets is first-order consistency property

Proof

let S be finite set of sentences of $L^{\mathbf{par}}$

- properties 1, 2, 3, 4: as in proof for propositional case
- ...

Proof (cont'd)

let S be finite set of sentences of $L^{\mathbf{par}}$

- property 5: let $\beta \in S$

  suppose neither $S \cup \{\beta_1\}$ nor $S \cup \{\beta_2\}$ is tableau consistent

  there exist closed tableaux T_1 for $S \cup \{\beta_1\}$ and T_2 for $S \cup \{\beta_2\}$

  without loss of generality: T_1 and T_2 do not share parameters

  tableau for $S = \{\beta, X_1, \ldots, X_n\}$:

$$\begin{array}{ccc} & \beta & \\ & X_1 & \\ & \vdots & \\ & X_n & \\ \beta_1 & & \beta_2 \\ T_1 & & T_2 \end{array}$$

  S is not tableau consistent

Proof (cont'd)

let S be finite set of sentences of $L^{\mathbf{par}}$

- property 6: let $\gamma \in S = \{\gamma, X_1, \ldots, X_n\}$

  suppose $S \cup \{\gamma(t)\}$ for some closed term t of $L^{\mathbf{par}}$ is not tableau consistent

  there exists closed tableau T for $S \cup \{\gamma(t)\}$ and hence also for S:

$$\begin{array}{cl} \gamma & \\ X_1 & \\ \vdots & \\ X_n & \\ \gamma(t) & \quad \text{apply } \gamma\text{-rule} \\ \text{rest of } T & \end{array}$$

  S is not tableau consistent

- property 7: similar

Theorem (Completeness for First-Order Tableaux)

every valid sentence X of L has tableau proof

Proof

- suppose X does not have tableau proof
- there is no closed tableau for $\{\neg X\}$
- $\{\neg X\}$ is tableau consistent
- $\{\neg X\}$ is satisfiable by First-Order Model Existence Theorem
- X cannot be valid

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Definition (Axiom Scheme 10)

$$\gamma \supset \gamma(t)$$

for any closed term t of $L^{\mathbf{par}}$

Definition (Universal Generalization)

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

provided p is parameter that does not occur in sentence $\Phi \supset \gamma$ and not in S in case of derivation from S

Definitions

- $S \vdash_{fh} X$ if there exists derivation of X from set S in first-order Hilbert systems
- if $\varnothing \vdash_{fh} X$ then X is theorem (and derivation is called proof)

Lemma

$$\frac{\gamma(p)}{\gamma} \qquad \textit{provided parameter } p \textit{ does not occur in sentence } \gamma$$

is derived rule in Hilbert system

Proof

suppose $\vdash_{fh} \gamma(p)$

1.	$\gamma(p)$	assumption
2.	$\gamma(p) \supset (\top \supset \gamma(p))$	Axiom Scheme 1
3.	$\top \supset \gamma(p)$	Modus Ponens
4.	$\top \supset \gamma$	Universal Generalization
5.	$(\top \supset \top) \supset \top$	Axiom Scheme 4
6.	$\top \supset \top$	Axiom Scheme 4
7.	$\top$	Modus Ponens
8.	γ	Modus Ponens

Example

$(\forall x)(P(x) \land Q(x)) \supset (\forall x)P(x)$ is theorem:

1.	$(\forall x)(P(x) \land Q(x)) \supset (P(p) \land Q(p))$	Axiom Scheme 10
2.	$(P(p) \land Q(p)) \supset P(p)$	Axiom Scheme 7
3.	$(\forall x)(P(x) \land Q(x)) \supset P(p)$	propositional logic
4.	$(\forall x)(P(x) \land Q(x)) \supset (\forall x)P(x)$	Universal Generalization

Theorem (Deduction Theorem)

in any first-order Hilbert System h with Modus Ponens and Universal Generalization as only rules of inference and at least Axiom Schemes 1 and 2:

$$S \cup \{X\} \vdash_{fh} Y \quad \Longleftrightarrow \quad S \vdash_{fh} X \supset Y$$

Proof (if direction)

- suppose $S \cup \{X\} \vdash_{fh} Y$
- let $\Pi_1 \colon Z_1, \dots, Z_n$ be derivation of Y from $S \cup \{X\}$, so $Z_n = Y$
- consider new sequence $\Pi_2 \colon X \supset Z_1, \dots, X \supset Z_n$
- insert extra lines into Π_2 as follows:

  1. $\cdots$
  2. $\cdots$
  3. $\cdots$
  4. if Z_i is derived with Universal Generalization from Z_j with $j < i$ then $Z_j = (\Phi \supset \gamma(p))$ and $Z_i = (\Phi \supset \gamma)$

     insert steps of (propositional) proof of $(X \wedge \Phi) \supset \gamma(p)$ from $X \supset Z_j$

     insert $(X \wedge \Phi) \supset \gamma$ (UG; p cannot occur in $(X \wedge \Phi) \supset \gamma$)

     insert steps of (propositional) proof of $X \supset Z_i$

     before $X \supset Z_i$

Example

$\{(\forall x)(P(x) \supset Q(x)), (\forall x)P(x)\} \vdash_{fh} (\forall x)Q(x)$:

1.	$(\forall x)P(x)$	
2.	$(\forall x)P(x) \supset P(p)$	Axiom Scheme 10
3.	$P(p)$	Modus Ponens
4.	$(\forall x)(P(x) \supset Q(x))$	
5.	$(\forall x)(P(x) \supset Q(x)) \supset (P(p) \supset Q(p))$	Axiom Scheme 10
6.	$P(p) \supset Q(p)$	Modus Ponens
7.	$Q(p)$	Modus Ponens
8.	$(\forall x)Q(x)$	Universal Generalization

Theorem (Strong Hilbert Soundness and Completeness)

for set S of sentences of L and sentence X of L:

$$S \vdash_{fh} X \iff S \vDash_f X$$

Lemma

$$\frac{\delta(p) \supset \Phi}{\delta \supset \Phi}$$ *provided parameter p does not occur in sentence* $\delta \supset \Phi$

is derived rule in Hilbert system

Proof

suppose $\vdash_{fh} \delta(p) \supset \Phi$

1.	$\delta(p) \supset \Phi$	assumption
2.	$(\delta(p) \supset \Phi) \supset (\neg\Phi \supset \neg\delta(p))$	propositional logic
3.	$\neg\Phi \supset \neg\delta(p)$	Modus Ponens
4.	$\neg\Phi \supset \neg\delta$	Universal Generalization
5.	$(\neg\Phi \supset \neg\delta) \supset (\delta \supset \Phi)$	propositional logic
6.	$\delta \supset \Phi$	Modus Ponens

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Theorem (Replacement Theorem)

given first-order formulas $\Phi(A)$, X, Y of language L and model $M = \langle \mathbf{D}, \mathbf{I} \rangle$ for L
if $X \equiv Y$ is true in M then $\Phi(X) \equiv \Phi(Y)$ is true in M

Proof

- $X^{\mathbf{I},\mathbf{A}} = Y^{\mathbf{I},\mathbf{A}}$ for every assignment $\mathbf{A}$
- $[\Phi(X)]^{\mathbf{I},\mathbf{A}} = [\Phi(Y)]^{\mathbf{I},\mathbf{A}}$ by structural induction on $\Phi(A)$:
  - atomic and propositional cases are straightforward
  - $\Phi(A) = (\forall y)\Psi(A)$

    $[\Psi(X)]^{\mathbf{I},\mathbf{A}} = [\Psi(Y)]^{\mathbf{I},\mathbf{A}}$ for every assignment $\mathbf{A}$ (induction hypothesis)

    ...

Proof (cont'd)

- $[\Phi(X)]^{\mathbf{I},\mathbf{A}} = [\Phi(Y)]^{\mathbf{I},\mathbf{A}}$ by structural induction on $\Phi(A)$:
  - $\Phi(A) = (\forall y)\Psi(A)$

    $[\Psi(X)]^{\mathbf{I},\mathbf{A}} = [\Psi(Y)]^{\mathbf{I},\mathbf{A}}$ for every assignment $\mathbf{A}$ (induction hypothesis)

    let $\mathbf{B}$ be arbitrary assignment

$$\begin{aligned}
[\Phi(X)]^{\mathbf{I},\mathbf{B}} = \mathrm{t} &\iff [\Psi(X)]^{\mathbf{I},\mathbf{A}} = \mathrm{t} \text{ for every } y\text{-variant } \mathbf{A} \text{ of } \mathbf{B} \\
&\iff [\Psi(Y)]^{\mathbf{I},\mathbf{A}} = \mathrm{t} \text{ for every } y\text{-variant } \mathbf{A} \text{ of } \mathbf{B} \\
&\iff [\Phi(Y)]^{\mathbf{I},\mathbf{B}} = \mathrm{t}
\end{aligned}$$

  - $\Phi(A) = (\exists y)\Psi(A)$ similar

Corollary

if $X \equiv Y$ is valid then $\Phi(X) \equiv \Phi(Y)$ is valid

Definition

all occurrences of atomic formula A in $\Phi(A)$ are positive provided

1. $\Phi(A) = A$
2. $\Phi(A) = \neg\neg\Psi(A)$ and all occurrences of A in $\Psi(A)$ are positive
3. $\Phi(A)$ is α-formula and all occurrences of A in α_1 and in α_2 are positive
4. $\Phi(A)$ is β-formula and all occurrences of A in β_1 and in β_2 are positive
5. $\Phi(A)$ is γ-formula with quantified variable x and all occurrences of A in $\gamma(x)$ are positive
6. $\Phi(A)$ is δ-formula with quantified variable x and all occurrences of A in $\delta(x)$ are positive

Example

$R(x, y)$ occurs positively in $(\forall x)[P(x, y) \supset \neg(\exists y)\neg R(x, y)]$:
$R(x, y)$ occurs positively in $P(x, y) \supset \neg(\exists y)\neg R(x, y)$

Theorem (Implicational Replacement Theorem)

given first-order formulas $\Phi(A)$, X, Y of language L and model $M = \langle \mathbf{D}, \mathbf{I} \rangle$ for L

if all occurrences of A in $\Phi(A)$ are positive and $X \supset Y$ is true in M then $\Phi(X) \supset \Phi(Y)$ is true in M

Definition

A has only negative occurrences in $\Phi(A)$ provided A has only positive occurrences in $\neg\Phi(A)$

Corollary

if all occurrences of A in $\Phi(A)$ are negative and $Y \supset X$ is true in M then $\Phi(X) \supset \Phi(Y)$ is true in M

Definition

- quantified subformula of formula Φ is essentially universal if it is positive subformula $(\forall x)\varphi$ or negative subformula $(\exists x)\varphi$
- quantified subformula of formula Φ is essentially existential if it is positive subformula $(\exists x)\varphi$ or negative subformula $(\forall x)\varphi$

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Lemma

given formula Ψ *with free variables among* $x, y_1, \ldots, y_n$ *and n-place function symbol* f *that does not occur in* Ψ*, for any model* $M = \langle \mathbf{D}, \mathbf{I} \rangle$ *there exist models* $N_1 = \langle \mathbf{D}, \mathbf{J}_1 \rangle$ *and* $N_2 = \langle \mathbf{D}, \mathbf{J}_2 \rangle$ *such that* $\mathbf{I}$, $\mathbf{J}_1$, $\mathbf{J}_2$ *differ only on interpretation of* f

- $(\exists x)\Psi \supset \Psi\{x/f(y_1, \ldots, y_n)\}$ *is true in* N_1
- $\Psi\{x/f(y_1, \ldots, y_n)\} \supset (\forall x)\Psi$ *is true in* N_2

Proof

given $d_1, \ldots, d_n \in \mathbf{D}$, we define $f^{\mathbf{J}_1}(d_1, \ldots, d_n)$ as follows:

- let $\mathbf{A}$ be assignment such that $y_1^{\mathbf{A}} = d_1$, $\ldots$, $y_n^{\mathbf{A}} = d_n$
- if $(\exists x)\Psi^{\mathbf{I},\mathbf{A}} = \mathsf{f}$ then $f^{\mathbf{J}_1}(d_1, \ldots, d_n) = d$ with d arbitrary member of D
- if $(\exists x)\Psi^{\mathbf{I},\mathbf{A}} = \mathsf{t}$ then $\Psi^{\mathbf{I},\mathbf{B}} = \mathsf{t}$ for some x-variant $\mathbf{B}$ of $\mathbf{A}$ and $f^{\mathbf{J}_1}(d_1, \ldots, d_n) = x^{\mathbf{B}}$ for one such $\mathbf{B}$

$(\exists x)\Psi \supset \Psi\{x/f(y_1, \ldots, y_n)\}$ is true in $N_1 = \langle \mathbf{D}, \mathbf{J}_1 \rangle$ by construction

Notation

$\Psi(x)$ for Ψ and $\Psi(f(y_1,\ldots,y_n))$ for $\Psi\{x/f(y_1,\ldots,y_n)\}$

Theorem (Skolemization)

given

- *formula* $\Psi(x)$ *with free variables* $x, y_1, \ldots, y_n$
- *formula* $\Phi(A)$ *such that* $\Phi((\exists x)\Psi(x))$ *is sentence*
- *n-place function symbol* f *that does not occur in* $\Phi((\exists x)\Psi(x))$

if all occurrences of A *in* $\Phi(A)$ *are*

1. *positive then* $\{\Phi((\exists x)\Psi(x))\}$ *is satisfiable*
   if and only if $\{\Phi(\Psi(f(y_1,\ldots,y_n)))\}$ *is satisfiable*
2. *negative then* $\{\Phi((\forall x)\Psi(x))\}$ *is satisfiable*
   if and only if $\{\Phi(\Psi(f(y_1,\ldots,y_n)))\}$ *is satisfiable*

Proof

$\Leftarrow$ suppose $\{\Phi(\Psi(f(y_1,\ldots,y_n)))\}$ is satisfiable

$\Psi(f(y_1,\ldots,y_n)) \supset (\exists x)\Psi(x)$ is valid

$\Phi(\Psi(f(y_1,\ldots,y_n))) \supset \Phi((\exists x)\Psi(x))$ is true in every model
by Implicational Replacement Theorem

$\{\Phi((\exists x)\Psi(x))\}$ is satisfiable

$\Rightarrow$ suppose $\Phi((\exists x)\Psi(x))$ is true in model $\mathbf{M} = \langle \mathbf{D}, \mathbf{I} \rangle$

there exists model $\mathbf{N} = \langle \mathbf{D}, \mathbf{J} \rangle$ in which $(\exists x)\Psi(x) \supset \Psi(f(y_1,\ldots,y_n))$ is true

$\Phi((\exists x)\Psi(x)) \supset \Phi(\Psi(f(y_1,\ldots,y_n)))$ is true in $\mathbf{N}$
by Implicational Replacement Theorem

$\Phi((\exists x)\Psi(x))$ is true in $\mathbf{N}$ (since it is true in $\mathbf{M}$ and $\mathbf{M}$ and $\mathbf{N}$ differ only on f)

$\Phi(\Psi(f(y_1,\ldots,y_n)))$ is true in $\mathbf{N}$

Skolemization

repeatedly replace

- positively occurring existentially quantified subformulas
- negatively occurring universally quantified subformulas

Lemma

if $\neg X'$ is Skolemized version of sentence $\neg X$ then
X is valid if and only if X' is valid

Proof

$$\begin{aligned} X \text{ is valid} &\iff \{\neg X\} \text{ is not satisfiable} \\ &\iff \{\neg X'\} \text{ is not satisfiable} \\ &\iff X' \text{ is valid} \end{aligned}$$

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Definition

sentence X' is validity functional form of X if $\neg X'$ is Skolemized version of $\neg X$

Lemma

validity functional form of sentence contains only essentially existential quantifiers

Definition

Herbrand universe of sentence X is set of all closed terms constructed from constant and function symbols of X

Example

- Herbrand universe of $(\forall x)[(\exists y)R(x,y) \supset R(b,f(x))]$ is set $\{b, f(b), f(f(b)), \dots\}$
- Herbrand universe of $(\forall x)(\exists y)R(x,y)$ is set $\{c_0\}$ (where c_0 is arbitrary new constant symbol)

Definitions

- Herbrand domain for sentence X is any finite non-empty subset of Herbrand universe of X
- given non-empty set $D = \{t_1, \ldots, t_n\}$ of closed terms and sentence X Herbrand expansion $\mathcal{E}(X, D)$ of X over D is defined recursively:
  1. if L is literal then $\mathcal{E}(L, D) = L$
  2. $\mathcal{E}(\neg\neg Z, D) = \mathcal{E}(Z, D)$
  3. $\mathcal{E}(\alpha, D) = \mathcal{E}(\alpha_1, D) \wedge \mathcal{E}(\alpha_2, D)$
  4. $\mathcal{E}(\beta, D) = \mathcal{E}(\beta_1, D) \vee \mathcal{E}(\beta_2, D)$
  5. $\mathcal{E}(\gamma, D) = \mathcal{E}(\gamma(t_1), D) \wedge \cdots \wedge \mathcal{E}(\gamma(t_n), D)$
  6. $\mathcal{E}(\delta, D) = \mathcal{E}(\delta(t_1), D) \vee \cdots \vee \mathcal{E}(\delta(t_n), D)$
- Herbrand expansion of X is Herbrand expansion of Y over D, where Y is validity functional form of X and D is any Herbrand domain for Y

Theorem (Herbrand's Theorem)

sentence X is valid if and only if some Herbrand expansion of X is tautology

Example

- sentence $(\forall z)(\exists w)(\forall x)[(\forall y)R(x,y) \supset R(w,z)]$ is valid
- validity functional form $(\exists w)[(\forall y)R(f(w),y) \supset R(w,c)]$ with Herbrand domain $D = \{c, f(c)\}$
- Herbrand expansion over D

$$\begin{aligned}\mathcal{E}((\exists w)[(\forall y)R(f(w),y) \supset R(w,c)], D)\\ = \neg R(f(c),c) \vee \neg R(f(c),f(c)) \vee R(c,c) \vee\\ \neg R(f(f(c)),c) \vee \neg R(f(f(c)),f(c)) \vee R(f(c),c)\end{aligned}$$

is tautology

Lemmata

given non-empty sets D and D' of closed terms such that $D \subseteq D'$

1. *for arbitrary sentence X, $\neg\mathcal{E}(X, D) \equiv \mathcal{E}(\neg X, D)$ is tautology*
2. *if X is sentence all of whose quantifiers are essentially existential then $\mathcal{E}(X, D) \supset \mathcal{E}(X, D')$ is tautology*
3. *if X is sentence all of whose quantifiers are essentially universal then $\mathcal{E}(X, D') \supset \mathcal{E}(X, D)$ is tautology*

Lemma

if all quantifiers in sentence X are essentially existential then $\mathcal{E}(X, D) \supset X$ is valid for any finite set D of closed terms

Proof

- suppose $D = \{t_1, \ldots, t_n\}$
- induction on X, interesting case: X is δ-formula
- induction hypothesis: $\mathcal{E}(\delta(t_i), D) \supset \delta(t_i)$ is valid for all $1 \leqslant i \leqslant n$
- $\mathcal{E}(\delta, D) = \mathcal{E}(\delta(t_1), D) \vee \cdots \vee \mathcal{E}(\delta(t_n), D)$
- $\mathcal{E}(\delta, D) \supset \delta(t_1) \vee \cdots \vee \delta(t_n)$ is valid
- $\mathcal{E}(\delta, D) \supset \delta$ is valid

Herbrand's Theorem (Soundness)

sentence X is valid if some Herbrand expansion of X is tautology

Definition

given first-order language L, set S of sentences of $L^{\textbf{par}}$ is Herbrand consistent if

1. S is finite
2. all members of S are essentially universal
3. $\neg\mathcal{E}(\bigwedge S, D)$ is no tautology, for any finite subset D of collection of all closed terms built from constant and function symbols of $L^{\textbf{par}}$

Lemma

collection of all Herbrand-consistent sets is first-order consistency property

Proof

let H be collection of all closed terms built from constant and function symbols of $L^{\textbf{par}}$ and let $\mathcal{C}$ be collection of all Herbrand-consistent sets

5. $S \in \mathcal{C}$, $\beta \in S$

   ...

Proof (cont'd)

5. $S \in \mathcal{C}$, $\beta \in S$

   suppose $S \cup \{\beta_1\} \notin \mathcal{C}$ and $S \cup \{\beta_2\} \notin \mathcal{C}$

   $\neg\mathcal{E}(\bigwedge S \wedge \beta_1, D_1)$ is tautology for some finite subset D_1 of H

   $\neg\mathcal{E}(\bigwedge S \wedge \beta_2, D_2)$ is tautology for some finite subset D_2 of H

   $\neg\mathcal{E}(\bigwedge S \wedge \beta_1, D)$ and $\neg\mathcal{E}(\bigwedge S \wedge \beta_2, D)$ are tautologies for $D = D_1 \cup D_2$

$$\begin{aligned}
\mathcal{E}(\textstyle\bigwedge S \wedge \beta, D) &= \mathcal{E}(\textstyle\bigwedge S, D) \wedge \mathcal{E}(\beta, D) \\
&= \mathcal{E}(\textstyle\bigwedge S, D) \wedge [\mathcal{E}(\beta_1, D) \vee \mathcal{E}(\beta_2, D)] \\
&\equiv [\mathcal{E}(\textstyle\bigwedge S, D) \wedge \mathcal{E}(\beta_1, D)] \vee [\mathcal{E}(\textstyle\bigwedge S, D) \wedge \mathcal{E}(\beta_2, D)] \\
&= \mathcal{E}(\textstyle\bigwedge S \wedge \beta_1, D) \vee \mathcal{E}(\textstyle\bigwedge S \wedge \beta_2, D) \\
\neg\mathcal{E}(\textstyle\bigwedge S \wedge \beta, D) &\equiv \neg\mathcal{E}(\textstyle\bigwedge S \wedge \beta_1, D) \wedge \neg\mathcal{E}(\textstyle\bigwedge S \wedge \beta_2, D) \text{ is tautology}
\end{aligned}$$

   hence $\beta \notin S$ $\lightning$

Proof (cont'd)

6. $S \in \mathcal{C}$, $\gamma \in S$

suppose $S \cup \{\gamma(t)\} \notin \mathcal{C}$ for some closed term t

$\neg\mathcal{E}(\bigwedge S \wedge \gamma(t), D)$ is tautology for some finite subset D of H

$\bigwedge S \wedge \gamma(t)$ is essentially universal and $D \subseteq D \cup \{t\}$

$\neg\mathcal{E}(\bigwedge S \wedge \gamma(t), D \cup \{t\})$ is tautology

$\mathcal{E}(\gamma, D \cup \{t\})$ is conjunction with $\mathcal{E}(\gamma(t), D \cup \{t\})$ as one of its conjuncts

$$\begin{aligned}
\mathcal{E}(\textstyle\bigwedge S \wedge \gamma, D \cup \{t\}) &= \mathcal{E}(\textstyle\bigwedge S, D \cup \{t\}) \wedge \mathcal{E}(\gamma, D \cup \{t\}) \\
&\supset \mathcal{E}(\textstyle\bigwedge S, D \cup \{t\}) \wedge \mathcal{E}(\gamma(t), D \cup \{t\}) \\
&= \mathcal{E}(\textstyle\bigwedge S \wedge \gamma(t), D \cup \{t\})
\end{aligned}$$

$\neg\mathcal{E}(\bigwedge S \wedge \gamma, D \cup \{t\})$ is tautology

hence $\gamma \notin S$ $\lightning$

Herbrand's Theorem (Completeness)

if sentence X whose quantifiers are all essentially existential is valid then $\mathcal{E}(X, D)$ is tautology for some Herbrand domain D for X

Proof

- let L be smallest first-order language in which X is sentence
- claim: $\mathcal{E}(X, D)$ is tautology for some finite set D of closed terms of $L^{\mathbf{par}}$

  let D be arbitrary finite set of closed terms of $L^{\mathbf{par}}$

  if $\mathcal{E}(X, D)$ is no tautology then $\{\neg X\}$ is Herbrand consistent:

  - $\{\neg X\}$ is finite
  - all quantifiers in $\neg X$ are essentially universal
  - $\neg\mathcal{E}(\neg X, D) \equiv \neg\neg\mathcal{E}(X, D) \equiv \mathcal{E}(X, D)$ is no tautology

  $\{\neg X\}$ is satisfiable in first-order model by Model Existence Theorem

  X is not valid $\lightning$

Herbrand's Theorem (Completeness)

if sentence X whose quantifiers are all essentially existential is valid then $\mathcal{E}(X, D)$ is tautology for some Herbrand domain D for X

Proof (cont'd)

- let L be smallest first-order language in which X is sentence
- $\mathcal{E}(X, D)$ is tautology for some finite set D of closed terms of L^{par}
- eliminate parameters from D by mapping each parameter p to some closed term $\tau(p)$ of L
- $\tau(\mathcal{E}(X, D)) = \mathcal{E}(X, \tau(D)) \equiv \mathcal{E}(X, D)$
- $\mathcal{E}(X, \tau(D))$ is tautology and $\tau(D)$ is Herbrand domain for X

Theorem (Herbrand's Theorem, Constructively)

there exists algorithm that extracts from tableau proof of first-order sentence X Herbrand expansion of X that is tautology

Proof (Herbrand's Theorem)

- without loss of generality: X is sentence in validity functional form
- let L be smallest language in which X is sentence
- let T be closed tableau for $\neg X$
- all quantifiers in $\neg X$ are essentially universal
- ...

Definition

tableau is parameter-free if it contains no parameter occurrences

Lemma

if all quantifiers of X are essentially existential then tableau proof of X can be converted into parameter-free tableau proof

Lemma

given

- *finite set S of sentences all of whose quantifiers are essentially universal*
- *closed parameter-free tableau T for S*

if D is set of closed terms that are used in applications of γ-rule in T then

$$\neg\mathcal{E}(\bigwedge S, D)$$

is tautology

Proof

- induction on number $\mathcal{B}(T,S)$ of nodes in T below initial nodes labeled with formulas in S
- base case: $\mathcal{B}(T,S) = 0$

  S contains contradiction and thus $\neg \bigwedge S$ is tautology
- induction step: $\mathcal{B}(T,S) > 0$

  case analysis on first rule application in T

$$\frac{\neg\neg Z}{Z} \qquad \frac{\neg\bot}{\top} \qquad \frac{\neg\top}{\bot} \qquad \frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta}{\beta_1 \mid \beta_2} \qquad \frac{\gamma}{\gamma(t)} \qquad \frac{\delta}{\delta(p)}$$

for any closed term t of L^{par} and new parameter p

Proof (cont'd)

- induction step: $\mathcal{B}(T, S) > 0$

  case analysis on first rule application in T: β-rule

  $S = \{Z_1, \dots, \beta, \dots, Z_k\}$

  T_1 is subtableau consisting of left half of T

  T_2 is subtableau consisting of right half of T

  D_1 is set of closed terms introduced by γ-rule in T_1

  D_2 is set of closed terms introduced by γ-rule in T_2

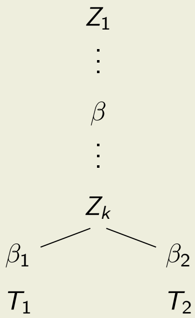
  $\mathcal{B}(T, S) = \mathcal{B}(T_1, S \cup \{\beta_1\}) + \mathcal{B}(T_1, S \cup \{\beta_1\}) + 2$

  $\neg\mathcal{E}(\bigwedge S \wedge \beta_1, D_1)$ and $\neg\mathcal{E}(\bigwedge S \wedge \beta_2, D_2)$ are tautologies (by IH)

  $\neg\mathcal{E}(\bigwedge S \wedge \beta_1, D)$ and $\neg\mathcal{E}(\bigwedge S \wedge \beta_2, D)$ are tautologies

  $\neg\mathcal{E}(\bigwedge S \wedge \beta, D) \equiv \neg\mathcal{E}(\bigwedge S \wedge \beta_1, D) \wedge \neg\mathcal{E}(\bigwedge S \wedge \beta_2, D)$ is tautology

  $\neg\mathcal{E}(\bigwedge S, D)$ is tautology



Proof (cont'd)

- induction step: $\mathcal{B}(T, S) > 0$

  case analysis on first rule application in T: γ-rule

  $S = \{Z_1, \dots, \gamma, \dots, Z_k\}$

  $\mathcal{B}(T, S) = \mathcal{B}(T, S \cup \{\gamma(t)\}) + 1$

  D is set of closed terms introduced by γ-rule in T considered as tableau for S

  D_0 is set of closed terms introduced by γ-rule in T considered as tableau for $S \cup \{\gamma(t)\}$

  $D = D_0 \cup \{t\}$

  $\neg\mathcal{E}(\bigwedge S \wedge \gamma(t), D_0)$ is tautology by induction hypothesis

  $\neg\mathcal{E}(\bigwedge S \wedge \gamma(t), D)$ is tautology

  $\neg\mathcal{E}(\bigwedge S \wedge \gamma, D) \equiv \neg\mathcal{E}(\bigwedge S, D)$ is tautology

$$
\begin{array}{c}
Z_1 \\
\vdots \\
\gamma \\
\vdots \\
Z_k \\
\gamma(t) \\
T
\end{array}
$$

Proof (Herbrand's Theorem, cont'd)

- without loss of generality: X is sentence in validity functional form
- let L be smallest language in which X is sentence
- let T be closed tableau for $\neg X$
- all quantifiers in $\neg X$ are essentially universal
- previous lemma (with $S = \{\neg X\}$): $\neg \mathcal{E}(\neg X, D)$ is tautology for set D of closed terms that are used in applications of γ-rule in T
- $\neg \mathcal{E}(\neg X, D) \equiv \mathcal{E}(X, D)$

Outline

- Overview of this Lecture
- First-Order Semantic Tableaux
- First-Order Hilbert Systems
- Replacement Theorem
- Skolemization
- Herbrand's Theorem
- Exercises
- Further Reading

Fitting

- Exercise 6.1.1 (you need to do half (5) of them; your choice) !
- Exercise 6.1.2
- Bonus Exercise 6.1.3
- Exercise 6.3.2
- Bonus Exercise 6.4.2
- Exercise 6.5.1 (for the same choice as in 6.1.1)
- Exercise 6.5.2
- Bonus Exercise 6.5.4 or Exercise 6.5.5
- Exercise 8.3.1
- Exercise 8.6.2 !
- Bonus Exercise 8.6.4
- Exercise 8.7.1 !

Outline

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Fitting

- Section 6.1 !
- Section 6.3 !
- Section 6.4 !
- Section 6.5 !
- Section 8.2
- Section 8.3 !
- Section 8.6 !
- Section 8.7 !