Computational Logic

Vincent van Oostrom
Course/slides by Aart Middeldorp

Department of Computer Science
University of Innsbruck

SS 2020

Outline

- Overview of this lecture
- Transforming Hilbert Style proof into tableau proof with cut
- Gentzen's Hauptsatz: Cut Elimination
- Craig's Interpolation Theorem
- Prenex Form
- Exercises
- Further Reading

Overview of this lecture

Last week we have seen Herbrand's theorem connects semantics to syntax by relating validity (being true in all models) of a first-order sentence X to an Herbrand expansion of X (a syntactic expansion yielding a sentence that is essentially propositional, obtained by instantiating quantified variables by closed terms from a finite Herbrand domain D) being a tautology. We gave two proofs, the first one based on Model Existence merely showing the existence of D, and the second one showing how to construct a suitable D (from certain closed terms appearing in a parameter-free tableau proof of X) and suitable Herbrand expansion. Herbrand's theorem allows to split proof search into two parts: searching for a suitable expansion and proving that indeed that is suitable, a tautology. In that way, Herbrand's theorem gives a handle on automated theorem proving. Such aspects are left to the follow-up course.

- Using the above we show a suitable Herbrand expansion can be constructed from Hilbert System proofs as well, in two steps:
  1. Hilbert System proofs can easily be transformed into tableau proofs, when extended with a new tableau expansion rule called cut.
  2. The cut expansion rule can be eliminated from tableau proofs (Gentzen's Hauptsatz), yielding a cut-free, i.e. ordinary, tableau proof.

- A cut can be thought of as using a lemma in a proof, so cut-elimination expresses that using lemmas can be avoided in principle. Its proof is based on the idea that each time a lemma is used it could be replaced by (an instance of) its proof. (The cut-elimination procedure will be inside–out, from the leaves toward the root.) However, this will be infeasible in practice, since copy-pasting of proofs will immediately lead to an exponential blow up of proof sizes when lemmas depend on other lemmas which depend on further lemmas etc.. . . .
- Craig's interpolation theorem is shown to hold for 1st order logic. Like for Herbrand's theorem we give both a non-constructive proof, based on Model Existence as in the propositional case, and a construction of an interpolant Z of $X \supset Y$ from a tableau proof of $X \supset Y$ by means of an inference system. The idea is to first construct interpolants for each of the branches, and then work our way upward from the leaves toward the root of the tableau, guided by the applied tableau expansion rules. To enable construction of interpolants, formulas inferred from X and Y in the tableau are labelled with L and R respectively (e.g. closing using formulas both inferred from X should yield a different interpolant, than when one was inferred from, say, X and the other from Y). The construction allows for a refinement due to Lyndon, stating that positive/negative predicates in Z occur positive/negatively in X, Y.

- We conclude with two transformations of 1st-order formulas, first into prenex form (a list of quantifiers followed by quantifier-free formula, its matrix) preserving equivalence, and next, by Skolemisation, into prenex form having only universal quantifiers preserving satisfiability.

  Just like in propositional logic one often preprocesses formulas (say into conjunctive or disjunctive or negation normal form) before applying a proof procedure (e.g. SAT solvers working on CNFs), in 1st order logic proof procedures may be (e.g. resolution) based on one or both of these transformations into (universal) prenex form. E.g. Skolemisation is often (but not here) presented for prenex forms only.

  Even if prenex forms simplify (the presentation of) such proof procedures, as it naturally brings about a decomposition into a propositional part (its matrix) and a 1st order part (its quantifiers), such transformations into some kind of normal form may, as in the propositional case, incur additional costs in actually proving. Such aspects are left to the follow-up course.

Part I: Propositional Logic

compactness, completeness, Hilbert systems, Hintikka's lemma, interpolation, logical consequence, model existence theorem, propositional semantic tableaux, soundness

Part II: First-Order Logic

compactness, completeness, Craig's interpolation theorem, cut elimination, first-order semantic tableaux, Herbrand models, Herbrand's theorem, Hilbert systems, Hintikka's lemma, Löwenheim–Skolem, logical consequence, model existence theorem, prenex form, skolemization, soundness

Part III: Limitations and Extensions of First-Order Logic

Curry-Howard isomorphism, intuitionistic logic, Kripke models, second-order logic, simply-typed λ-calculus, (simply-typed) combinatory logic

Part I: Propositional Logic

compactness, completeness, Hilbert systems, Hintikka's lemma, interpolation, logical consequence, model existence theorem, propositional semantic tableaux, soundness

Part II: First-Order Logic

compactness, completeness, Craig's interpolation theorem, cut elimination, first-order semantic tableaux, Herbrand models, Herbrand's theorem, Hilbert systems, Hintikka's lemma, Löwenheim–Skolem, logical consequence, model existence theorem, prenex form, skolemization, soundness

Part III: Limitations and Extensions of First-Order Logic

Curry-Howard isomorphism, intuitionistic logic, Kripke models, second-order logic, simply-typed λ-calculus, (simply-typed) combinatory logic

Outline

- Overview of this lecture
- Transforming Hilbert Style proof into tableau proof with cut
- Gentzen's Hauptsatz: Cut Elimination
- Craig's Interpolation Theorem
- Prenex Form
- Exercises
- Further Reading

Question

How to obtain tautologous Herbrand expansion from proof in Hilbert system?

Question

How to obtain tautologous Herbrand expansion from proof in Hilbert system?

Answer

transform Hilbert system proof into tableau proof with cut, and then use earlier result for tableau

Question

How to obtain tautologous Herbrand expansion from proof in Hilbert system?

Answer

transform Hilbert system proof into tableau proof with cut, and then use earlier result for tableau

Question

How to obtain tautologous Herbrand expansion from proof in Hilbert system?

Answer

transform Hilbert system proof into tableau proof with cut, and then use earlier result for tableau

Proof

- Hilbert system axioms are easy

Question

How to obtain tautologous Herbrand expansion from proof in Hilbert system?

Answer

transform Hilbert system proof into tableau proof with cut, and then use earlier result for tableau

Proof

- Hilbert system axioms are easy
- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

$$\neg(\Phi \supset \gamma(p))$$

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

$$\begin{array}{l} \neg(\Phi \supset \gamma(p)) \\ \quad \Phi \\ \neg\gamma(p) \end{array}$$

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

$$\begin{array}{c} \neg(\Phi \supset \gamma(p)) \\ \Phi \\ \neg\gamma(p) \\ T \end{array}$$

Proof (cont'd)

- Universal Generalization Rule

  $$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

  where p is parameter that does not occur in sentence $\Phi \supset \gamma$

  assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

  $$\begin{array}{c} \neg(\Phi \supset \gamma(p)) \\ \Phi \\ \neg\gamma(p) \\ T \end{array}$$

  we may assume that formula $\neg(\Phi \supset \gamma(p))$ on first line is not used again

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

$$\begin{array}{c} \Phi \\ \neg\gamma(p) \\ T \end{array}$$

we may assume that formula $\neg(\Phi \supset \gamma(p))$ on first line is not used again

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

$$\neg(\Phi \supset (\forall x)\varphi(x))$$

$$\Phi$$

$$\neg\gamma(p)$$

$$T$$

we may assume that formula $\neg(\Phi \supset \gamma(p))$ on first line is not used again

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

$$\begin{array}{c} \Phi \\ \neg\gamma(p) \\ T \end{array} \qquad\qquad \begin{array}{l} \neg(\Phi \supset (\forall x)\varphi(x)) \\ \;\Phi \\ \neg(\forall x)\varphi(x) \end{array}$$

we may assume that formula $\neg(\Phi \supset \gamma(p))$ on first line is not used again

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

$$\begin{array}{ccc} & \qquad\qquad & \neg(\Phi \supset (\forall x)\varphi(x)) \\ & & \Phi \\ \Phi & & \neg(\forall x)\varphi(x) \\ \neg\gamma(p) & & \neg\gamma(p) \\ T & & \end{array}$$

we may assume that formula $\neg(\Phi \supset \gamma(p))$ on first line is not used again

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

$$\begin{array}{ccc}
 & \qquad\qquad & \neg(\Phi \supset (\forall x)\varphi(x)) \\
 & & \Phi \\
\Phi & & \neg(\forall x)\varphi(x) \\
\neg\gamma(p) & & \neg\gamma(p) \\
T & & T
\end{array}$$

we may assume that formula $\neg(\Phi \supset \gamma(p))$ on first line is not used again

Proof (cont'd)

- Universal Generalization Rule

$$\frac{\Phi \supset \gamma(p)}{\Phi \supset \gamma}$$

where p is parameter that does not occur in sentence $\Phi \supset \gamma$

assume $\gamma = (\forall x)\varphi(x)$ and consider tableau proof of $\Phi \supset \gamma(p)$:

$$\begin{array}{ccc} & & \neg(\Phi \supset (\forall x)\varphi(x)) \\ & & \Phi \\ \Phi & & \neg(\forall x)\varphi(x) \\ \neg\gamma(p) & & \neg\gamma(p) \\ T & & T \end{array}$$

we may assume that formula $\neg(\Phi \supset \gamma(p))$ on first line is not used again

tableau proof of $\Phi \supset \gamma$

Proof (cont'd)

- Modus Ponens

$$\frac{X \qquad X \supset Y}{Y}$$

Proof (cont'd)

- Modus Ponens

$$\frac{X \qquad X \supset Y}{Y}$$

complicated

Proof (cont'd)

- Modus Ponens

$$\frac{X \qquad X \supset Y}{Y}$$

complicated; introduce Tableau Cut Rule

$$\frac{}{X \mid \neg X}$$

Proof (cont'd)

- Modus Ponens

  $$\frac{X \qquad X \supset Y}{Y}$$

  complicated; introduce Tableau Cut Rule

  $$\frac{}{X \mid \neg X}$$

  assume T_1 is tableau proof of X and T_2 is tableau proof of $X \supset Y$

Proof (cont'd)

- Modus Ponens

  $$\frac{X \qquad X \supset Y}{Y}$$

  complicated; introduce Tableau Cut Rule

  $$\frac{}{X \mid \neg X}$$

  assume T_1 is tableau proof of X and T_2 is tableau proof of $X \supset Y$

  tableau proof of Y:

  $$\neg Y$$

Proof (cont'd)

- Modus Ponens

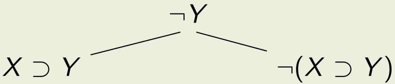
  $$\frac{X \qquad X \supset Y}{Y}$$

  complicated; introduce Tableau Cut Rule

  $$\frac{}{X \mid \neg X}$$

  assume T_1 is tableau proof of X and T_2 is tableau proof of $X \supset Y$

  tableau proof of Y:



$\neg Y$

$X \supset Y$

$\neg(X \supset Y)$

Proof (cont'd)

- Modus Ponens

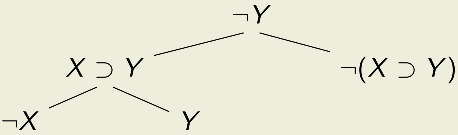
$$\frac{X \qquad X \supset Y}{Y}$$

complicated; introduce Tableau Cut Rule

$$\frac{}{X \mid \neg X}$$

assume T_1 is tableau proof of X and T_2 is tableau proof of $X \supset Y$

tableau proof of Y:



Proof (cont'd)

- Modus Ponens

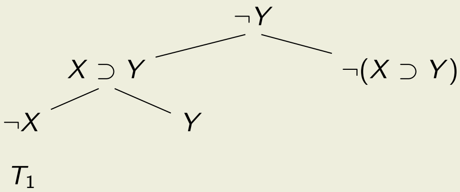
$$\frac{X \qquad X \supset Y}{Y}$$

complicated; introduce Tableau Cut Rule

$$\frac{}{X \mid \neg X}$$

assume T_1 is tableau proof of X and T_2 is tableau proof of $X \supset Y$

tableau proof of Y:



$\neg Y$

$X \supset Y$

$\neg(X \supset Y)$

$\neg X$

Y

T_1

Proof (cont'd)

- Modus Ponens

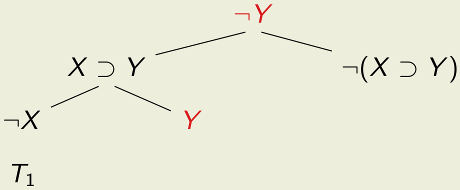
  $$\frac{X \qquad X \supset Y}{Y}$$

  complicated; introduce Tableau Cut Rule

  $$\frac{}{X \mid \neg X}$$

  assume T_1 is tableau proof of X and T_2 is tableau proof of $X \supset Y$

  tableau proof of Y:



Proof (cont'd)

- Modus Ponens

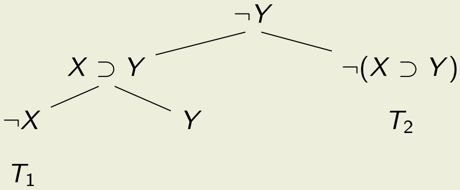
$$\frac{X \qquad X \supset Y}{Y}$$

complicated; introduce Tableau Cut Rule

$$\frac{}{X \mid \neg X}$$

assume T_1 is tableau proof of X and T_2 is tableau proof of $X \supset Y$

tableau proof of Y:



Outline

- Overview of this lecture
- Transforming Hilbert Style proof into tableau proof with cut
- Gentzen's Hauptsatz: Cut Elimination
- Craig's Interpolation Theorem
- Prenex Form
- Exercises
- Further Reading

Theorem (Cut Elimination)

any closed tableau with applications of Cut Rule can be converted into closed tableau without

Theorem (Cut Elimination)

any closed tableau with applications of Cut Rule can be converted into closed tableau without

Fact 1

if $X = (A \circ B)$ with primary connective $\circ$ then

- $\{X, \neg X\}$ consists of α-formula and β-formula

Theorem (Cut Elimination)

any closed tableau with applications of Cut Rule can be converted into closed tableau without

Fact 1

if $X = (A \circ B)$ with primary connective $\circ$ then

- $\{X, \neg X\}$ consists of α-formula and β-formula
- one of α_1 and β_1 is negation of other
- one of α_2 and β_2 is negation of other

Theorem (Cut Elimination)

any closed tableau with applications of Cut Rule can be converted into closed tableau without

Fact 1

if $X = (A \circ B)$ with primary connective $\circ$ then

- $\{X, \neg X\}$ consists of α-formula and β-formula
- one of α_1 and β_1 is negation of other
- one of α_2 and β_2 is negation of other

Fact 2

if $X = (Qx)\varphi(x)$ with $Q \in \{\forall, \exists\}$ then

- $\{X, \neg X\}$ consists of γ-formula and δ-formula

Theorem (Cut Elimination)

any closed tableau with applications of Cut Rule can be converted into closed tableau without

Fact 1

if $X = (A \circ B)$ with primary connective $\circ$ then

- $\{X, \neg X\}$ consists of α-formula and β-formula
- one of α_1 and β_1 is negation of other
- one of α_2 and β_2 is negation of other

Fact 2

if $X = (Qx)\varphi(x)$ with $Q \in \{\forall, \exists\}$ then

- $\{X, \neg X\}$ consists of γ-formula and δ-formula
- one of $\gamma(t)$ and $\delta(t)$ is negation of other

Definitions

given cut to sentences X and $\neg X$ in tableau T

Definitions

given cut to sentences X and $\neg X$ in tableau T

- cut is at branch end if there are no sentences below X or no below $\neg X$

Definitions

given cut to sentences X and $\neg X$ in tableau T

- cut is at branch end if there are no sentences below X or no below $\neg X$
- rank of cut is rank of X

Definitions

given cut to sentences X and $\neg X$ in tableau T

- cut is at branch end if there are no sentences below X or no below $\neg X$
- rank of cut is rank of X
- weight of cut is number of sentences below X and $\neg X$

Definitions

given cut to sentences X and $\neg X$ in tableau T

- cut is at branch end if there are no sentences below X or no below $\neg X$
- rank of cut is rank of X
- weight of cut is number of sentences below X and $\neg X$
- cut is minimal if there are no cuts below it in T

Definitions

given cut to sentences X and $\neg X$ in tableau T

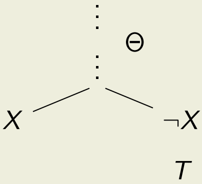
- cut is at branch end if there are no sentences below X or no below $\neg X$
- rank of cut is rank of X
- weight of cut is number of sentences below X and $\neg X$
- cut is minimal if there are no cuts below it in T

Lemma

closed tableau T with cut at branch end can be transformed into closed tableau in which cut is eliminated

Proof

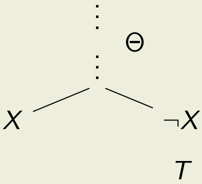
consider cut at branch end



two cases

Proof

consider cut at branch end

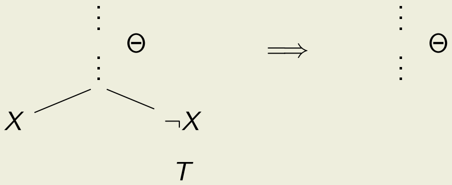


two cases

1. X plays no role in closure of left branch

Proof

consider cut at branch end

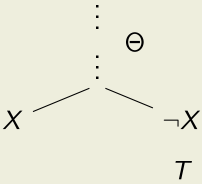


two cases

1. X plays no role in closure of left branch then Θ must be closed

Proof

consider cut at branch end

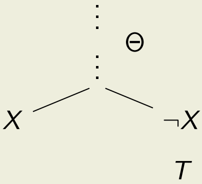


two cases

1. X plays no role in closure of left branch then Θ must be closed
2. X plays role in closure of left branch

Proof

consider cut at branch end

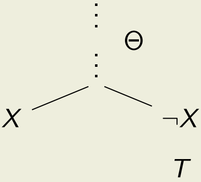


two cases

1. X plays no role in closure of left branch then Θ must be closed
2. X plays role in closure of left branch
    - if $X = \bot$ then $\neg X = \neg\bot$

Proof

consider cut at branch end



Θ

X $\neg X$

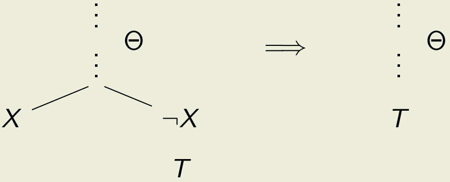
T

two cases

1. X plays no role in closure of left branch then Θ must be closed
2. X plays role in closure of left branch
    - if $X = \bot$ then $\neg X = \neg\bot$ plays no role in closure of right branch

Proof

consider cut at branch end

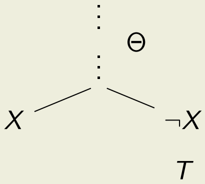


two cases

1. X plays no role in closure of left branch then Θ must be closed
2. X plays role in closure of left branch
    - if $X = \bot$ then $\neg X = \neg\bot$ plays no role in closure of right branch

Proof

consider cut at branch end

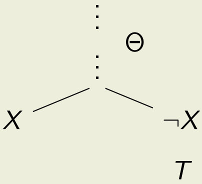


two cases

1. X plays no role in closure of left branch then Θ must be closed
2. X plays role in closure of left branch
    - if $X = \bot$ then $\neg X = \neg\bot$ plays no role in closure of right branch
    - if $X = (A \circ B)$ or $X = (\forall x)\varphi$ or $X = (\exists x)\varphi$ or X is atomic

Proof

consider cut at branch end



Θ

X $\neg X$

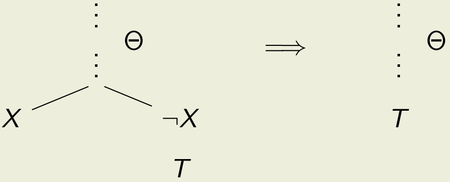
T

two cases

1. X plays no role in closure of left branch then Θ must be closed
2. X plays role in closure of left branch
    - if $X = \bot$ then $\neg X = \neg\bot$ plays no role in closure of right branch
    - if $X = (A \circ B)$ or $X = (\forall x)\varphi$ or $X = (\exists x)\varphi$ or X is atomic then $\neg X$ occurs in Θ

Proof

consider cut at branch end

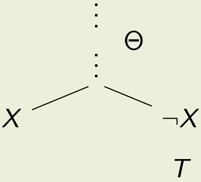


two cases

1. X plays no role in closure of left branch then Θ must be closed
2. X plays role in closure of left branch
    - if $X = \bot$ then $\neg X = \neg\bot$ plays no role in closure of right branch
    - if $X = (A \circ B)$ or $X = (\forall x)\varphi$ or $X = (\exists x)\varphi$ or X is atomic then $\neg X$ occurs in Θ

Proof

consider cut at branch end

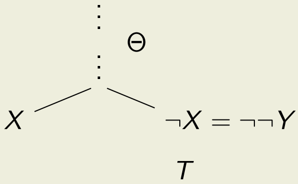


two cases

1. X plays no role in closure of left branch then Θ must be closed
2. X plays role in closure of left branch
    - if $X = \bot$ then $\neg X = \neg\bot$ plays no role in closure of right branch
    - if $X = (A \circ B)$ or $X = (\forall x)\varphi$ or $X = (\exists x)\varphi$ or X is atomic then $\neg X$ occurs in Θ
    - $X = \neg Y$ for some sentence Y

Proof (cont'd)

consider cut at branch end



Θ

X

$\neg X = \neg\neg Y$

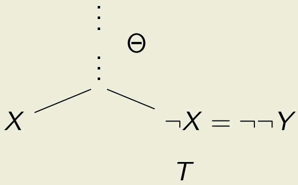
T

two cases

2. $X = \neg Y$ plays role in closure of left branch

Proof (cont'd)

consider cut at branch end



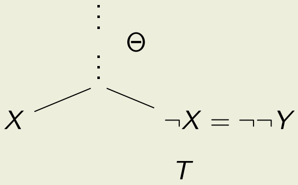
two cases

2. $X = \neg Y$ plays role in closure of left branch

   Y or $\neg\neg Y$ occurs in Θ

Proof (cont'd)

consider cut at branch end



Θ

X

$\neg X = \neg\neg Y$

T

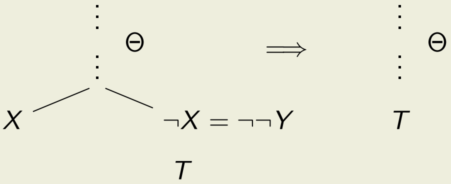
two cases

2. $X = \neg Y$ plays role in closure of left branch

   Y or $\neg\neg Y$ occurs in Θ

Proof (cont'd)

consider cut at branch end



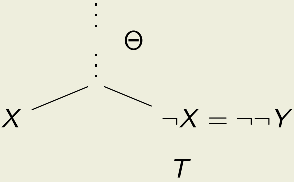
two cases

2. $X = \neg Y$ plays role in closure of left branch

   Y or $\neg\neg Y$ occurs in Θ

Proof (cont'd)

consider cut at branch end



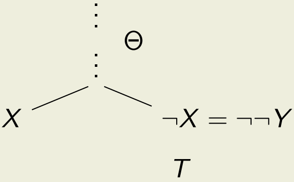
two cases

2. $X = \neg Y$ plays role in closure of left branch

   Y or $\neg\neg Y$ occurs in Θ

Proof (cont'd)

consider cut at branch end



two cases

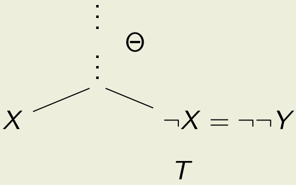
2. $X = \neg Y$ plays role in closure of left branch

   Y or $\neg\neg Y$ occurs in Θ

   $\neg\neg Y$ is used in right fork (for otherwise cut can be eliminated)

Proof (cont'd)

consider cut at branch end



two cases

2. $X = \neg Y$ plays role in closure of left branch

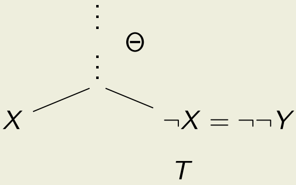
   Y or $\neg\neg Y$ occurs in Θ

   $\neg\neg Y$ is used in right fork (for otherwise cut can be eliminated)

   - applications of double negation rule applied to $\neg\neg Y$ can be dropped

Proof (cont'd)

consider cut at branch end



two cases

2. $X = \neg Y$ plays role in closure of left branch

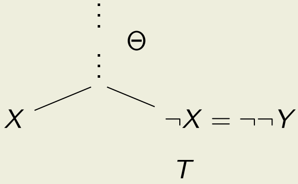
   Y or $\neg\neg Y$ occurs in Θ

   $\neg\neg Y$ is used in right fork (for otherwise cut can be eliminated)

   - applications of double negation rule applied to $\neg\neg Y$ can be dropped
   - if $\neg\neg Y$ is directly involved in closure of branch in right fork
     then $\neg Y$ or $\neg\neg\neg Y$ must occur in that branch

Proof (cont'd)

consider cut at branch end



two cases

2. $X = \neg Y$ plays role in closure of left branch

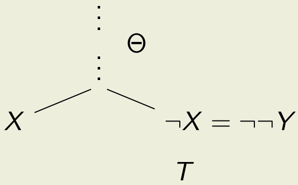
   Y or $\neg\neg Y$ occurs in Θ

   $\neg\neg Y$ is used in right fork (for otherwise cut can be eliminated)

   - applications of double negation rule applied to $\neg\neg Y$ can be dropped
   - if $\neg\neg Y$ is directly involved in closure of branch in right fork then $\neg Y$ or $\neg\neg\neg Y$ must occur in that branch

Proof (cont'd)

consider cut at branch end



two cases

2. $X = \neg Y$ plays role in closure of left branch

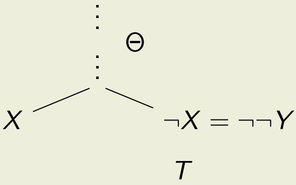
   Y or $\neg\neg Y$ occurs in Θ

   $\neg\neg Y$ is used in right fork (for otherwise cut can be eliminated)

   - applications of double negation rule applied to $\neg\neg Y$ can be dropped
   - if $\neg\neg Y$ is directly involved in closure of branch in right fork then $\neg Y$ or $\neg\neg\neg Y$ must occur in that branch

Proof (cont'd)

consider cut at branch end



two cases

2. $X = \neg Y$ plays role in closure of left branch

   Y or $\neg\neg Y$ occurs in Θ

   $\neg\neg Y$ is used in right fork (for otherwise cut can be eliminated)

   - applications of double negation rule applied to $\neg\neg Y$ can be dropped
   - if $\neg\neg Y$ is directly involved in closure of branch in right fork
     then $\neg Y$ or $\neg\neg\neg Y$ must occur in that branch (...)

Example

propositional tableau

$$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$$

Example

propositional tableau

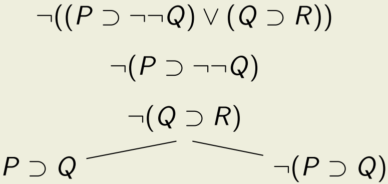
$$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$$

$$\neg(P \supset \neg\neg Q)$$

$$\neg(Q \supset R)$$

Example

propositional tableau



$$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$$

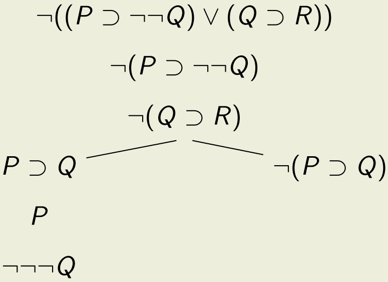
$$\neg(P \supset \neg\neg Q)$$

$$\neg(Q \supset R)$$

$$P \supset Q \qquad \neg(P \supset Q)$$

Example

propositional tableau



$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$

$\neg(P \supset \neg\neg Q)$

$\neg(Q \supset R)$

$P \supset Q$

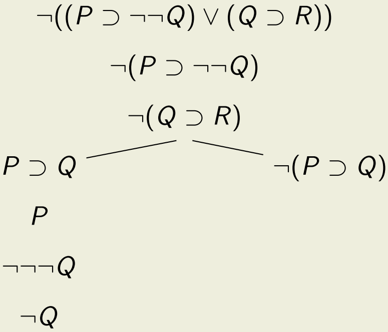
$\neg(P \supset Q)$

P

$\neg\neg\neg Q$

Example

propositional tableau



$$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$$

$$\neg(P \supset \neg\neg Q)$$

$$\neg(Q \supset R)$$

$$P \supset Q \qquad \neg(P \supset Q)$$

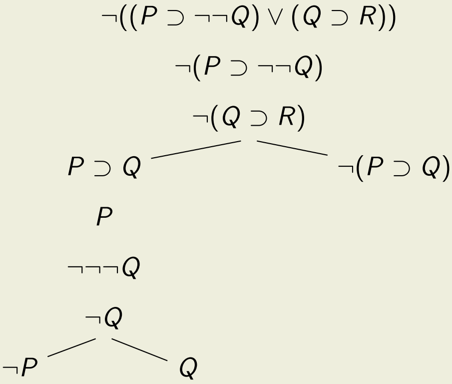
$$P$$

$$\neg\neg\neg Q$$

$$\neg Q$$

Example

propositional tableau



$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$

$\neg(P \supset \neg\neg Q)$

$\neg(Q \supset R)$

$P \supset Q$

$\neg(P \supset Q)$

P

$\neg\neg\neg Q$

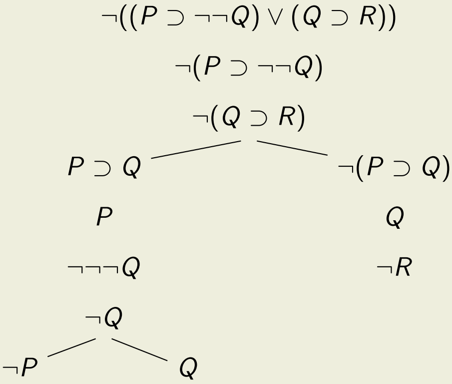
$\neg Q$

$\neg P$

Q

Example

propositional tableau



$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$

$\neg(P \supset \neg\neg Q)$

$\neg(Q \supset R)$

$P \supset Q$ $\quad$ $\neg(P \supset Q)$

P $\quad$ Q

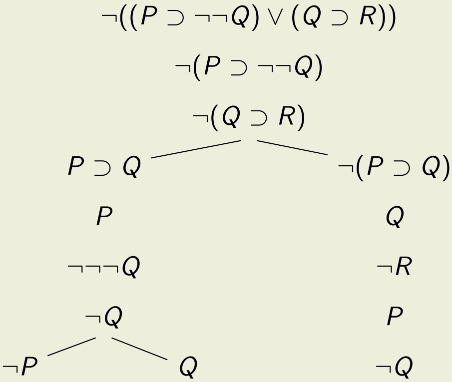
$\neg\neg\neg Q$ $\quad$ $\neg R$

$\neg Q$

$\neg P$ $\quad$ Q

Example

propositional tableau



$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$

$\neg(P \supset \neg\neg Q)$

$\neg(Q \supset R)$

$P \supset Q$

P

$\neg\neg\neg Q$

$\neg Q$

$\neg P$

Q

$\neg(P \supset Q)$

Q

$\neg R$

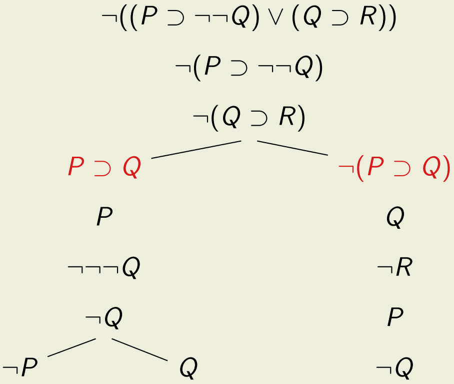
P

$\neg Q$

Example

propositional tableau

(minimal) cut



$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$

$\neg(P \supset \neg\neg Q)$

$\neg(Q \supset R)$

$P \supset Q$ | $\neg(P \supset Q)$

P | Q

$\neg\neg\neg Q$ | $\neg R$

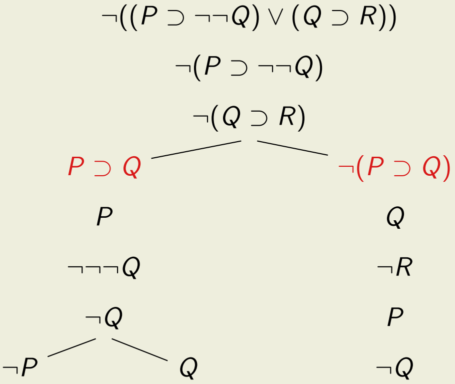
$\neg Q$ | P

$\neg P$ | Q | $\neg Q$

Example

propositional tableau

(minimal) cut
rank 1



$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$

$\neg(P \supset \neg\neg Q)$

$\neg(Q \supset R)$

Left branch:

$P \supset Q$

P

$\neg\neg\neg Q$

$\neg Q$

$\neg P$ | Q

Right branch:

$\neg(P \supset Q)$

Q

$\neg R$

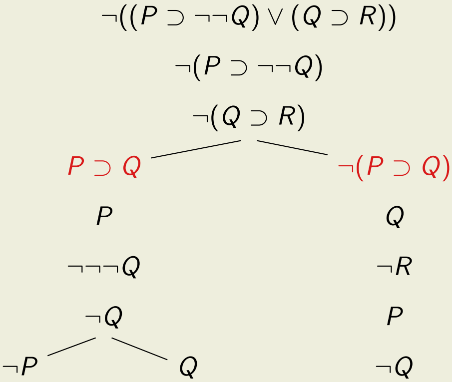
P

$\neg Q$

Example

propositional tableau

(minimal) cut
rank 1
weight 9



$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$

$\neg(P \supset \neg\neg Q)$

$\neg(Q \supset R)$

$P \supset Q$ $\qquad$ $\neg(P \supset Q)$

P $\qquad$ Q

$\neg\neg\neg Q$ $\qquad$ $\neg R$

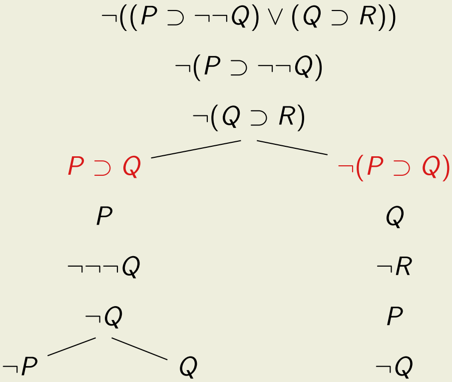
$\neg Q$ $\qquad$ P

$\neg P$ $\quad$ Q $\qquad$ $\neg Q$

Example

propositional tableau

(minimal) cut
rank 1
weight 9



$\neg((P \supset \neg\neg Q) \vee (Q \supset R))$

$\neg(P \supset \neg\neg Q)$

$\neg(Q \supset R)$

$P \supset Q$ | $\neg(P \supset Q)$

P | Q

$\neg\neg\neg Q$ | $\neg R$

$\neg Q$ | P

$\neg P$ | Q | $\neg Q$

Lemma (Key Lemma)

closed tableau T with minimal cut not at branch end of rank n and weight k can be transformed into closed tableau in which cut is replaced by cuts of lower rank or same rank but lower weight

Fact 3

if T is closed tableau for finite set S of sentences and $S \subseteq S'$ then there exists closed tableau for S' with same number of steps as T

Fact 3

if T is closed tableau for finite set S of sentences and $S \subseteq S'$ then there exists closed tableau for S' with same number of steps as T

Fact 4

if T is closed tableau for finite set $S \cup \{\delta(c)\}$ of sentences with parameter c that does not occur in S or δ then there exists closed tableau for $S \cup \{\delta(t)\}$ with same number of steps as T, for every closed term t

Fact 3

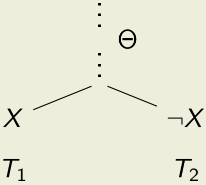
if T is closed tableau for finite set S of sentences and $S \subseteq S'$ then there exists closed tableau for S' with same number of steps as T

Fact 4

if T is closed tableau for finite set $S \cup \{\delta(c)\}$ of sentences with parameter c that does not occur in S or δ then there exists closed tableau for $S \cup \{\delta(t)\}$ with same number of steps as T, for every closed term t

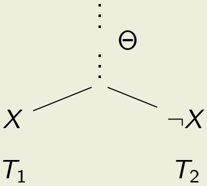
Proof of Key Lemma

consider minimal cut in tableau T



Proof of Key Lemma

consider minimal cut in tableau T

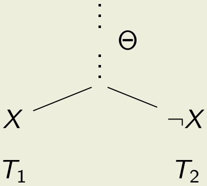


two cases

1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

Proof of Key Lemma

consider minimal cut in tableau T

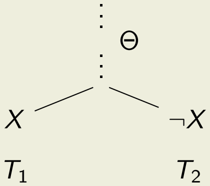


two cases

1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ
2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

Proof of Key Lemma

consider minimal cut in tableau T



Θ

X $\neg X$

T_1 T_2

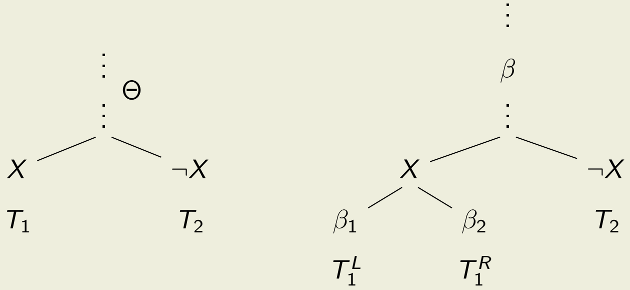
two cases

1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   β-case

Proof of Key Lemma

consider minimal cut in tableau T



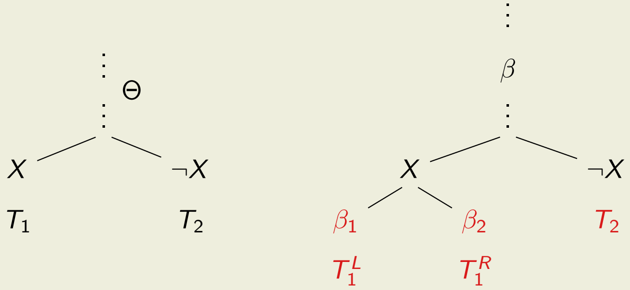
two cases

1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   β-case

Proof of Key Lemma

consider minimal cut in tableau T



two cases

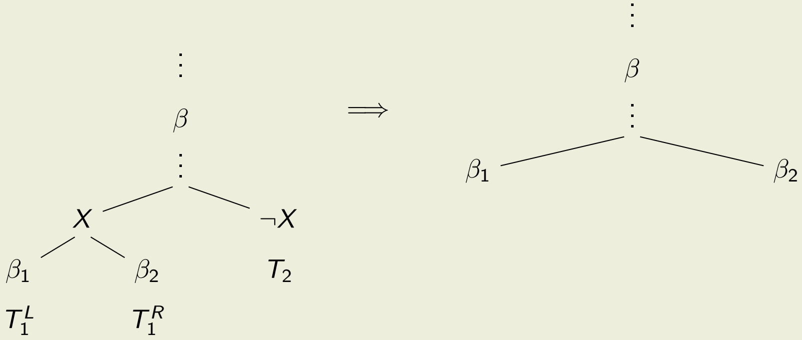
1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   β-case

   weight of cut is $|T_1^L| + |T_1^R| + |T_2| + 2$

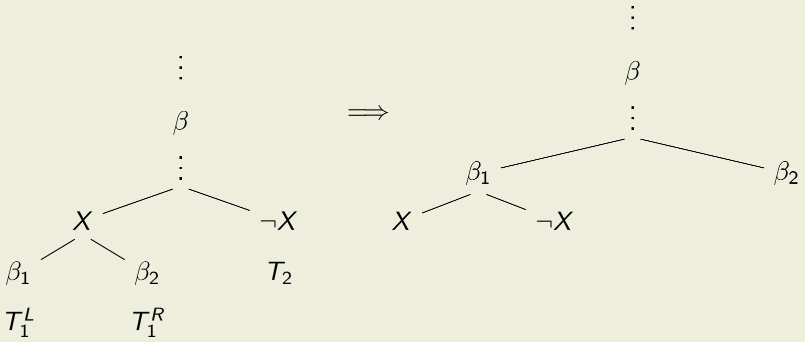
Proof of Key Lemma (cont'd)

modify T as follows:



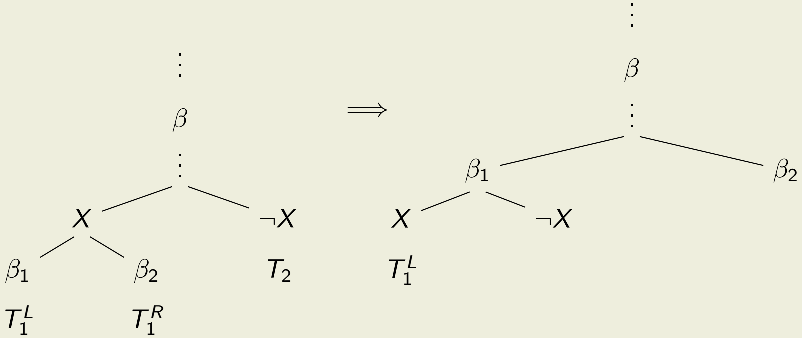
Proof of Key Lemma (cont'd)

modify T as follows:



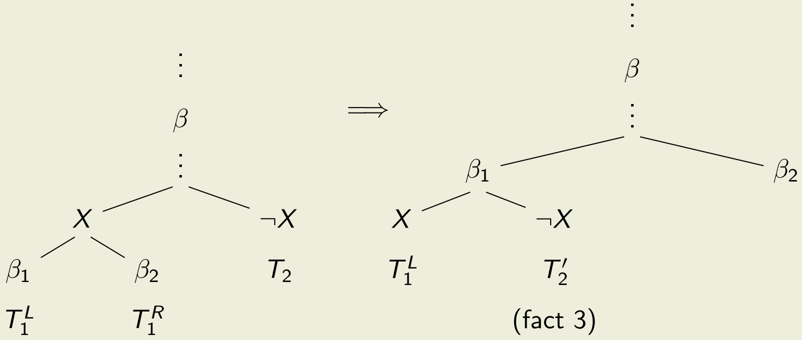
Proof of Key Lemma (cont'd)

modify T as follows:



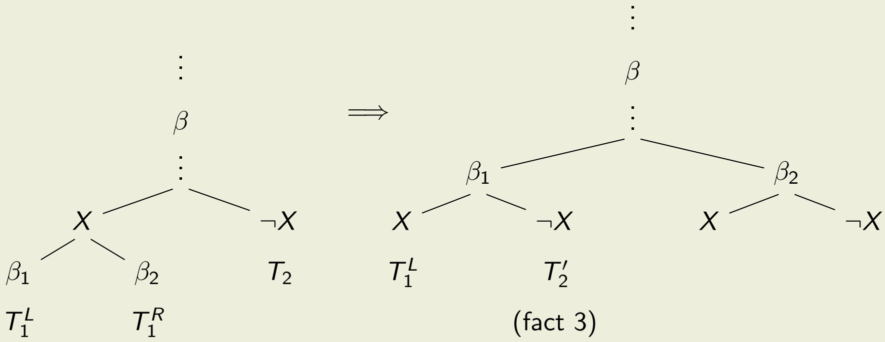
Proof of Key Lemma (cont'd)

modify T as follows:



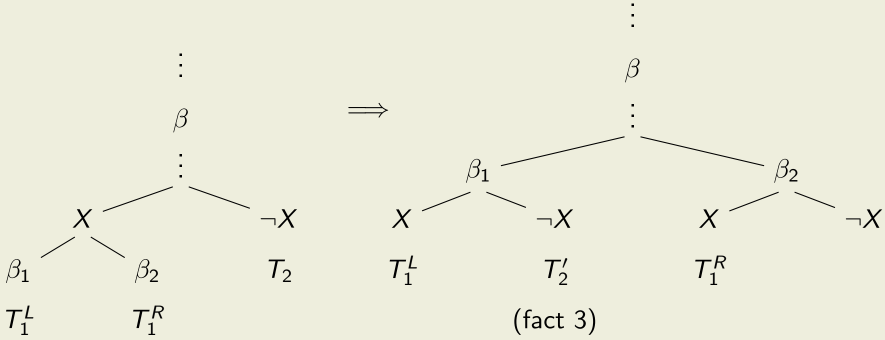
Proof of Key Lemma (cont'd)

modify T as follows:



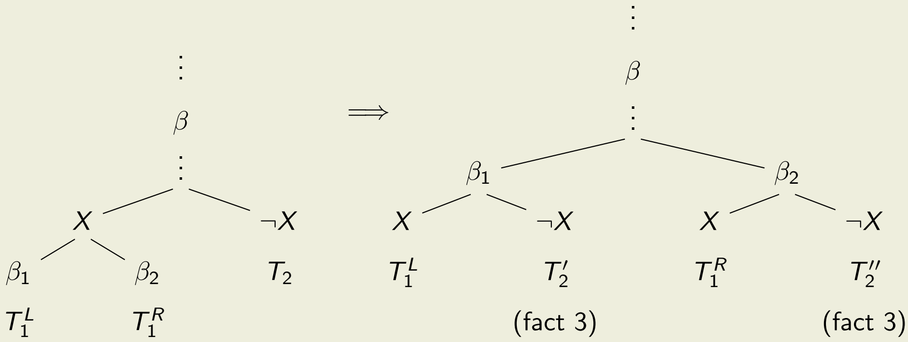
Proof of Key Lemma (cont'd)

modify T as follows:



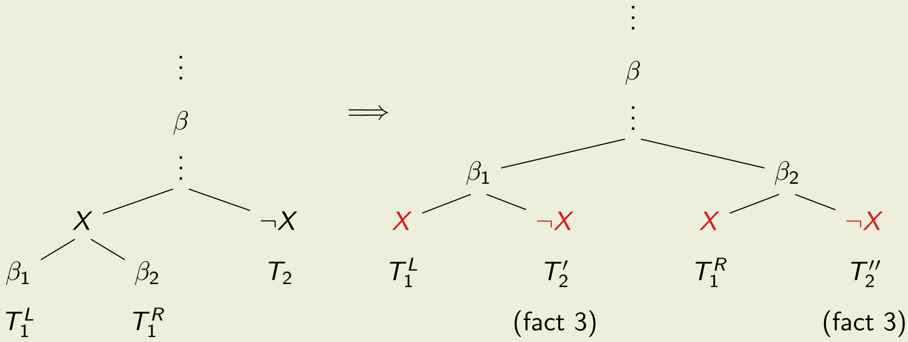
Proof of Key Lemma (cont'd)

modify T as follows:



Proof of Key Lemma (cont'd)

modify T as follows:



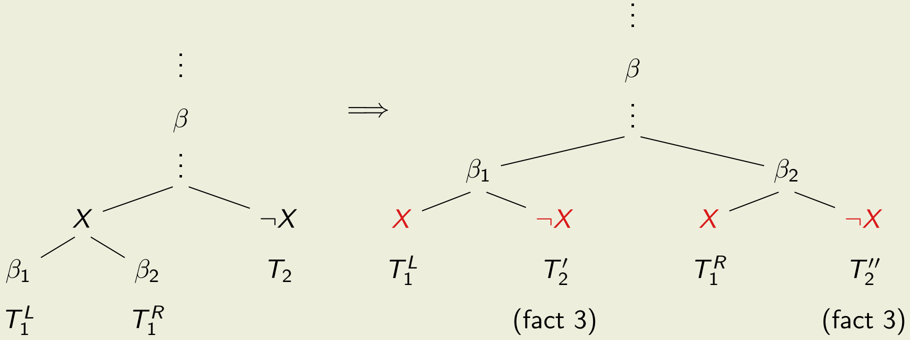
new cuts have weights

$$|T_1^L| + |T_2'|$$

$$|T_1^R| + |T_2''|$$

Proof of Key Lemma (cont'd)

modify T as follows:

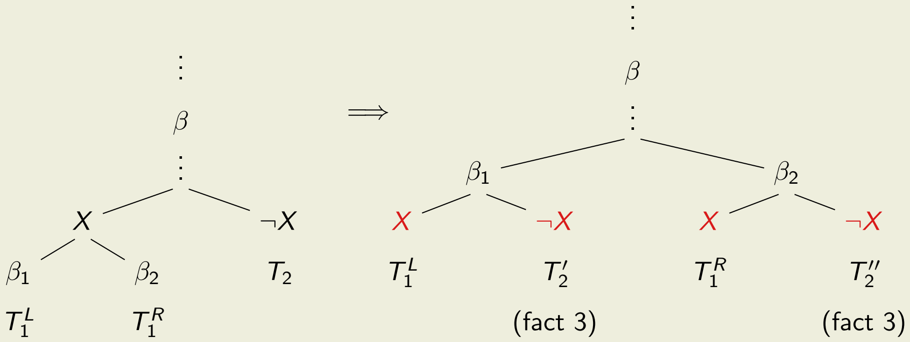


new cuts have weights

$$|T_1^L| + |T_2'| \;=\; |T_1^L| + |T_2|$$
$$|T_1^R| + |T_2''| \;=\; |T_1^R| + |T_2|$$

Proof of Key Lemma (cont'd)

modify T as follows:

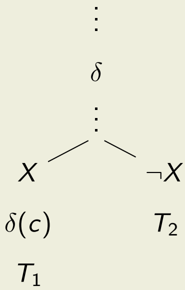


new cuts have weights

$$|T_1^L| + |T_2'| \;=\; |T_1^L| + |T_2| \;<\; |T_1^L| + |T_1^R| + |T_2| + 2$$
$$|T_1^R| + |T_2''| \;=\; |T_1^R| + |T_2| \;<\; |T_1^L| + |T_1^R| + |T_2| + 2$$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



$\vdots$

δ

$\vdots$

$X \qquad \neg X$

$\delta(c) \qquad T_2$

T_1

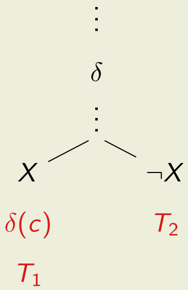
two cases

1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   δ-case

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



$\vdots$

δ

$\vdots$

X $\neg X$

$\delta(c)$ T_2

T_1

two cases

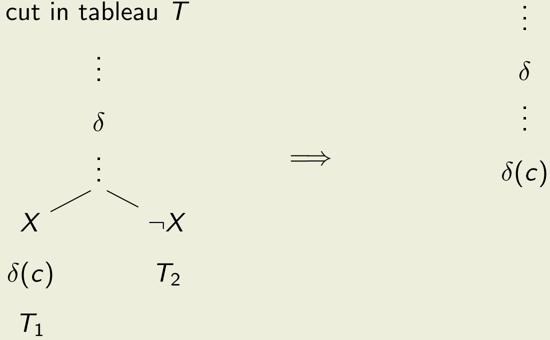
1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   δ-case

   weight of cut is $|T_1| + |T_2| + 1$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



$\vdots$

δ

$\vdots$

X $\neg X$

$\delta(c)$ T_2

T_1

$\Longrightarrow$

$\vdots$

δ

$\vdots$

$\delta(c)$

two cases

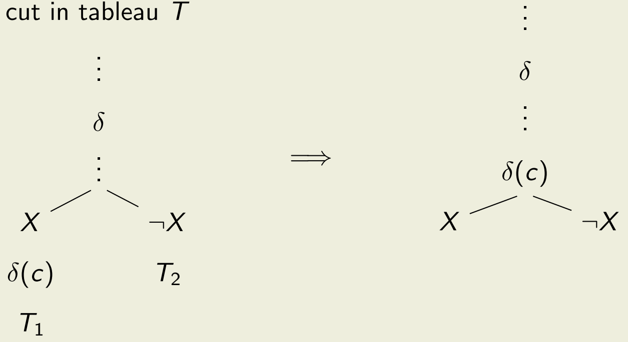
1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   δ-case

   weight of cut is $|T_1| + |T_2| + 1$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

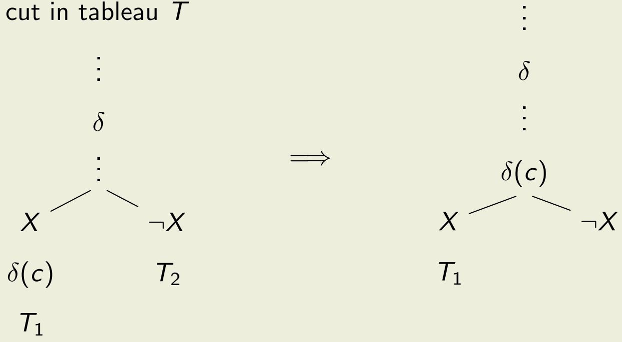
1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   δ-case

   weight of cut is $|T_1| + |T_2| + 1$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

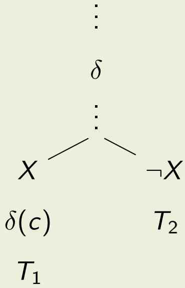
1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   δ-case

   weight of cut is $|T_1| + |T_2| + 1$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



$\vdots$

δ

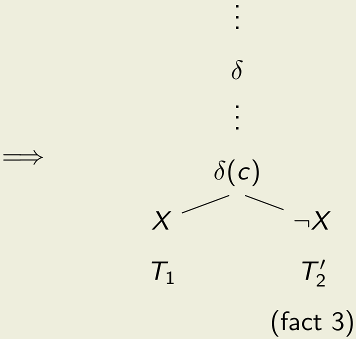
$\vdots$

X $\neg X$

$\delta(c)$ T_2

T_1

$\Longrightarrow$



$\vdots$

δ

$\vdots$

$\delta(c)$

X $\neg X$

T_1 T_2'

(fact 3)

two cases

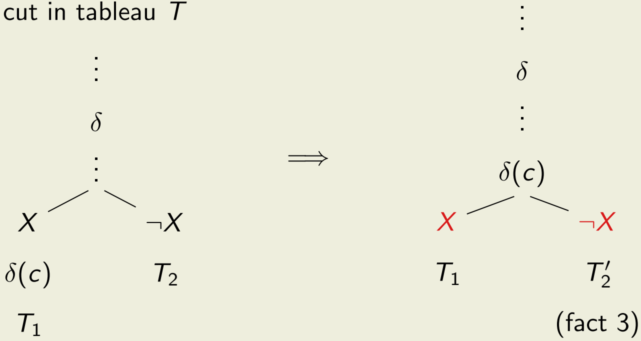
1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   δ-case

   weight of cut is $|T_1| + |T_2| + 1$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

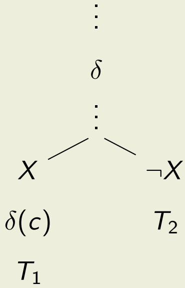
1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

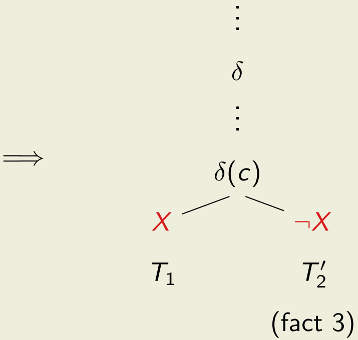
   δ-case

   weight of cut is $|T_1| + |T_2| + 1$
   weight of new cut is $|T_1| + |T_2'|$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T





two cases

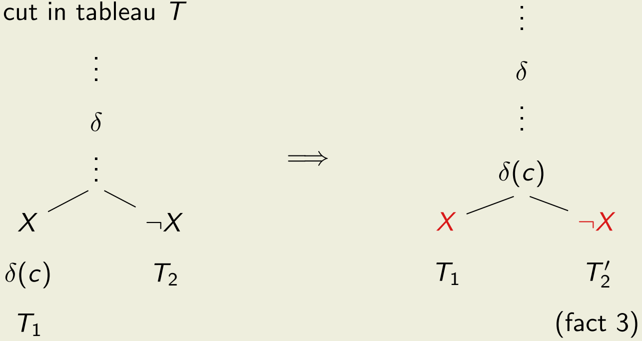
1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   δ-case

   weight of cut is $|T_1| + |T_2| + 1$
   weight of new cut is $|T_1| + |T_2'| = |T_1| + |T_2|$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

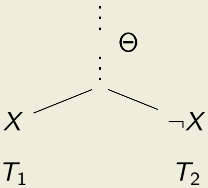
1. uppermost sentence in T_1 or T_2 was obtained by applying tableau rule to sentence from Θ

   δ-case

   weight of cut is $|T_1| + |T_2| + 1$
   weight of new cut is $|T_1| + |T_2'| = |T_1| + |T_2| < |T_1| + |T_2| + 1$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

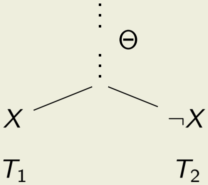


two cases

2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



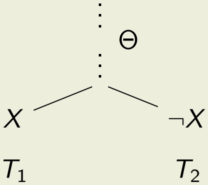
two cases

2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

   primary connective case: $X = A \circ B$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

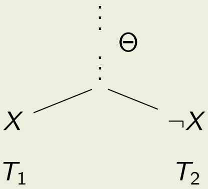
2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

   primary connective case: $X = A \circ B$

   suppose X is α-formula

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

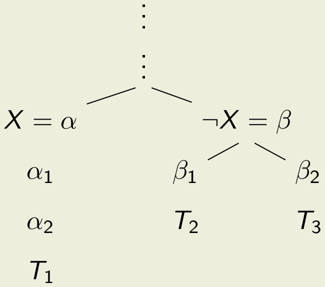
2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

   primary connective case: $X = A \circ B$

   suppose X is α-formula so $\neg X$ is β-formula (fact 1)

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



$X = \alpha$

α_1

α_2

T_1

$\neg X = \beta$

β_1 β_2

T_2 T_3

two cases

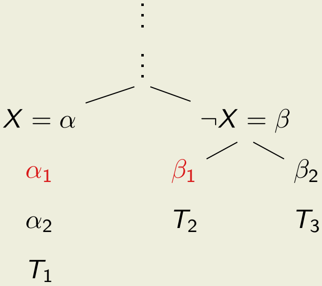
2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

   primary connective case: $X = A \circ B$

   suppose X is α-formula so $\neg X$ is β-formula (fact 1)

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

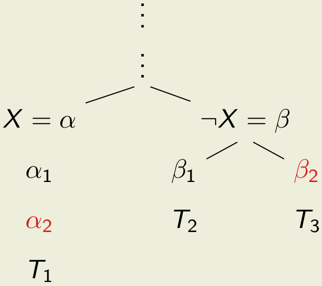
   primary connective case: $X = A \circ B$

   suppose X is α-formula so $\neg X$ is β-formula (fact 1)

   one of $\{\alpha_1, \beta_1\}$ is negation of other (fact 1)

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

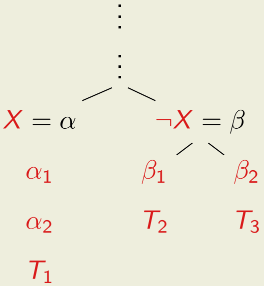
   primary connective case: $X = A \circ B$

   suppose X is α-formula so $\neg X$ is β-formula (fact 1)

   one of $\{\alpha_2, \beta_2\}$ is negation of other (fact 1)

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

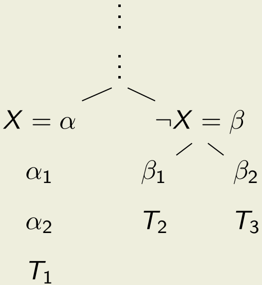


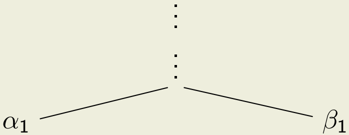
primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



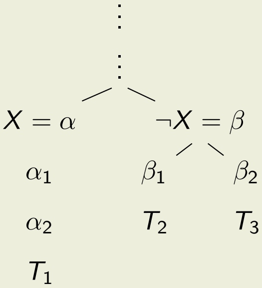


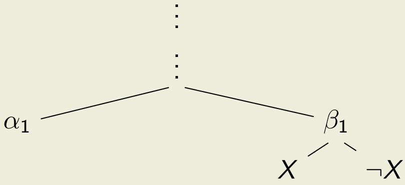
primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



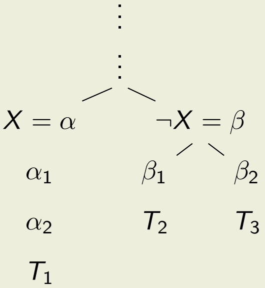


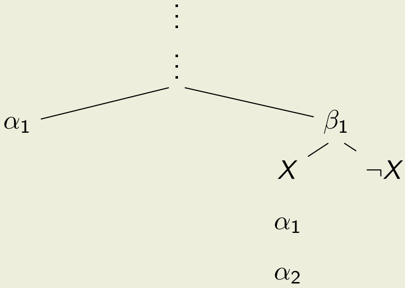
primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



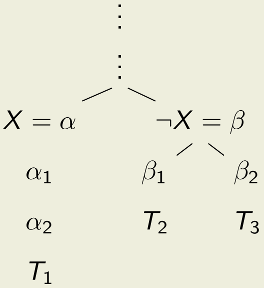


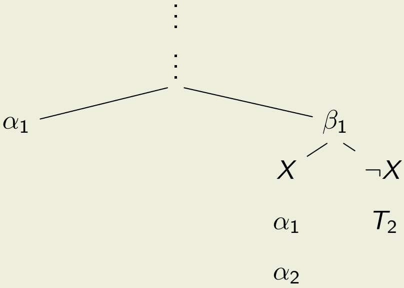
primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



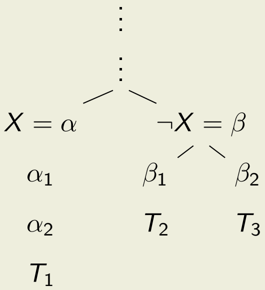


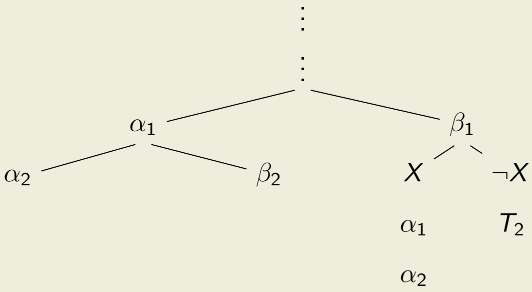
primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



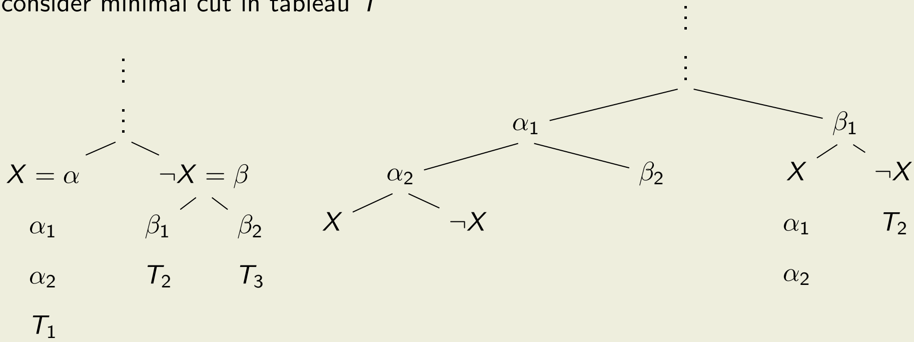


primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

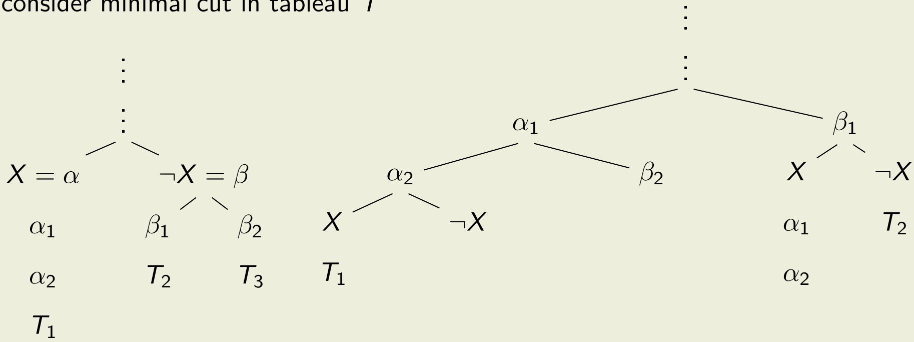


primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

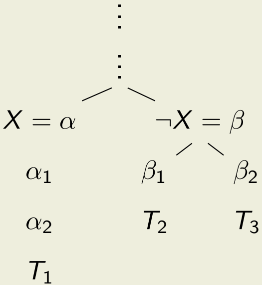


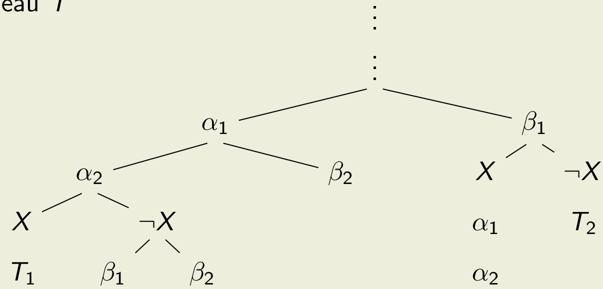
primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



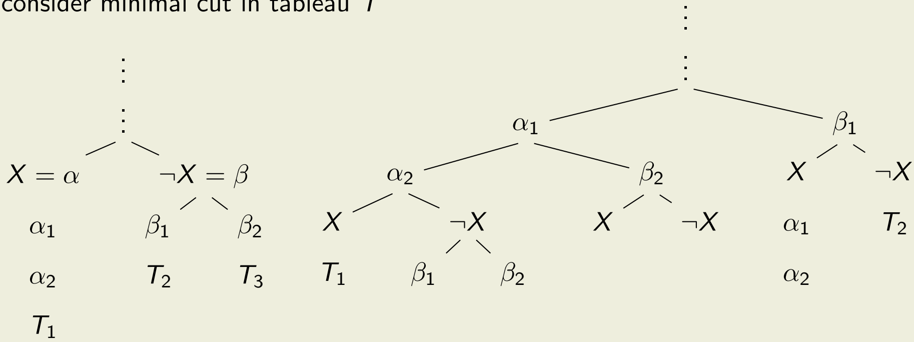


primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

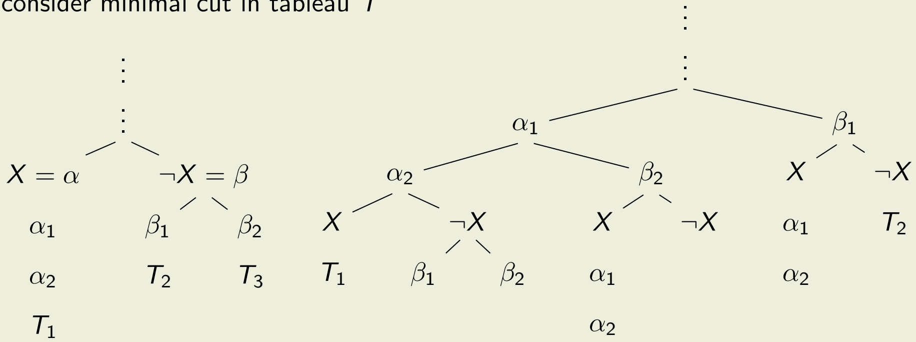


primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

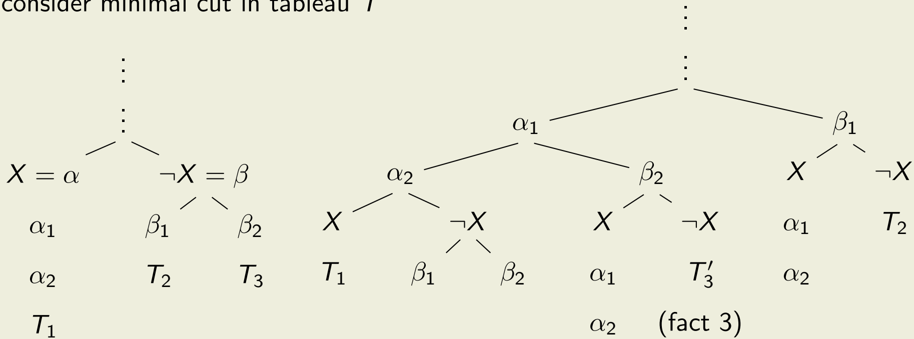


primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

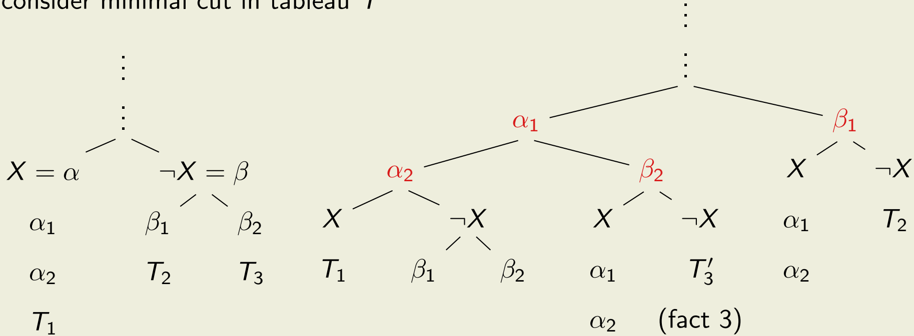


primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



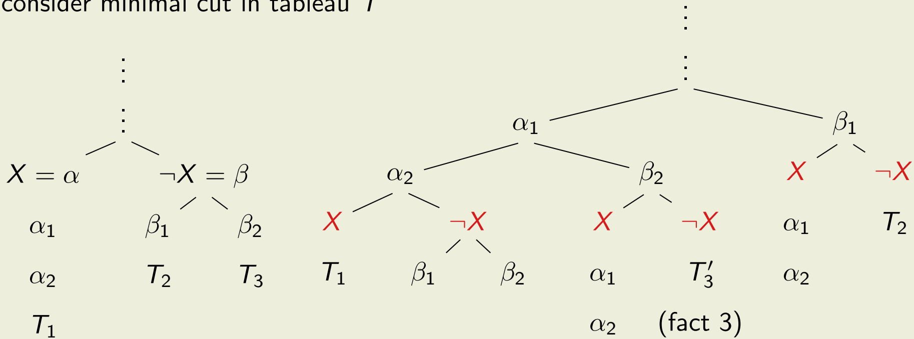
primary connective case: $X = \alpha$ and $\neg X = \beta$

weight of cut is $|T_1| + |T_2| + |T_3| + 4$

rank of cuts $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$ is smaller than rank of original cut $\{X, \neg X\}$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



primary connective case: $X = \alpha$ and $\neg X = \beta$

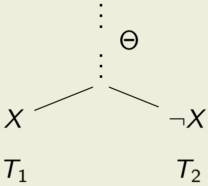
weight of cut is $|T_1| + |T_2| + |T_3| + 4$

rank of cuts $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$ is smaller than rank of original cut $\{X, \neg X\}$

weight of new cuts $\{X, \neg X\}$ is smaller than $|T_1| + |T_2| + |T_3| + 4$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



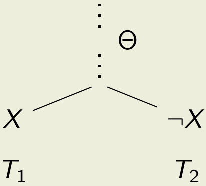
two cases

2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

   quantifier case

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

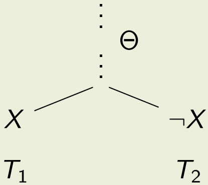
2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

   quantifier case

   suppose X is γ-formula

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



Θ

X $\quad$ $\neg X$

T_1 $\quad$ T_2

two cases

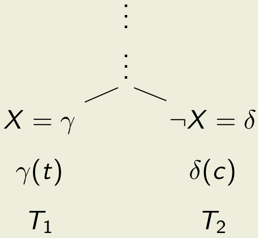
2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

   quantifier case

   suppose X is γ-formula so $\neg X$ is δ-formula (fact 2)

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



$X = \gamma$ $\qquad$ $\neg X = \delta$

$\gamma(t)$ $\qquad$ $\delta(c)$

T_1 $\qquad$ T_2

two cases

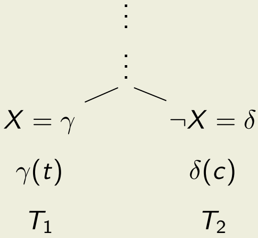
2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

   quantifier case

   suppose X is γ-formula so $\neg X$ is δ-formula (fact 2)

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



$X = \gamma$ $\quad$ $\neg X = \delta$

$\gamma(t)$ $\quad$ $\delta(c)$

T_1 $\quad$ T_2

two cases

2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

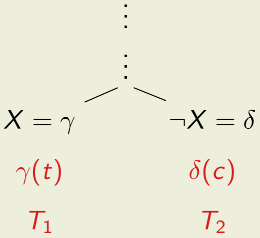
   quantifier case

   suppose X is γ-formula so $\neg X$ is δ-formula (fact 2)

   one of $\{\gamma(t), \delta(t)\}$ is negation of other (fact 2)

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



two cases

2. uppermost sentences in T_1 and T_2 were obtained by applying tableau rules to X and $\neg X$

   quantifier case

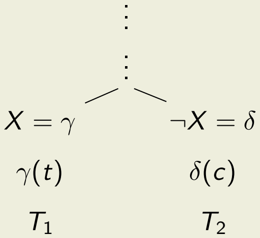
   suppose X is γ-formula so $\neg X$ is δ-formula (fact 2)

   one of $\{\gamma(t), \delta(t)\}$ is negation of other (fact 2)

   rank of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



$X = \gamma$ | $\neg X = \delta$

$\gamma(t)$ | $\delta(c)$

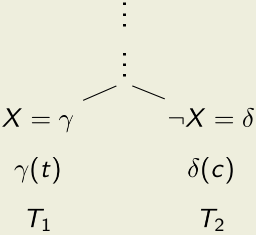
T_1 | T_2

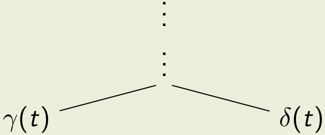
quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



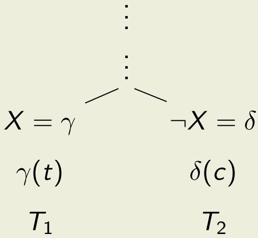


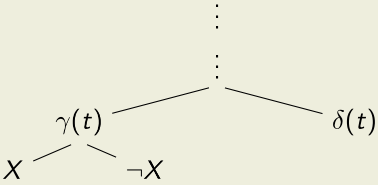
quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



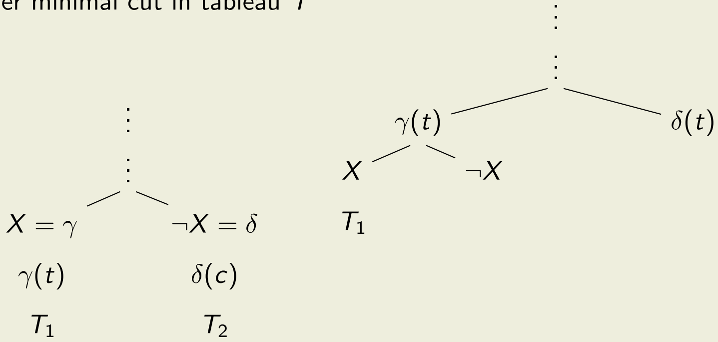


quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

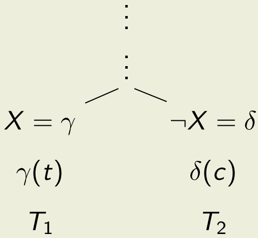


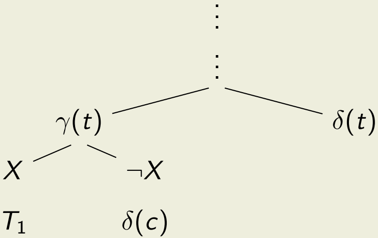
quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



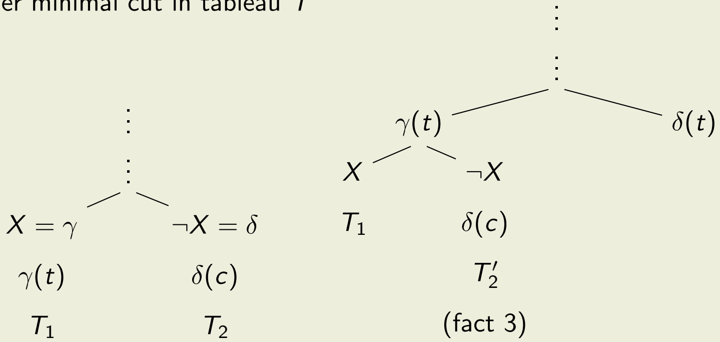


quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



$X = \gamma$

$\gamma(t)$

T_1

$\neg X = \delta$

$\delta(c)$

T_2

$\gamma(t)$

X

T_1

$\neg X$

$\delta(c)$

T_2'

(fact 3)

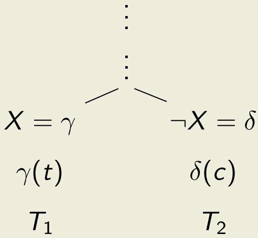
$\delta(t)$

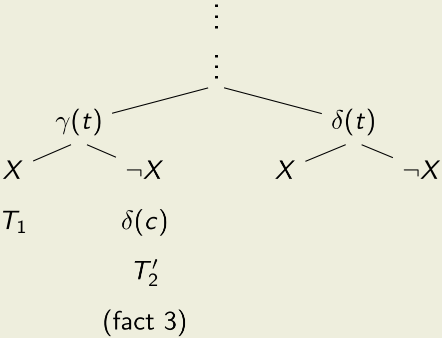
quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



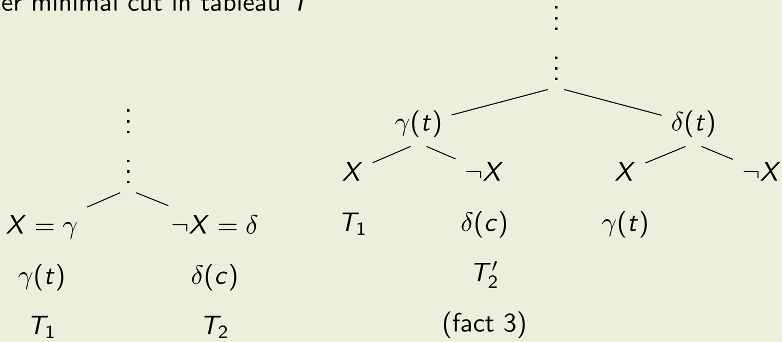


quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

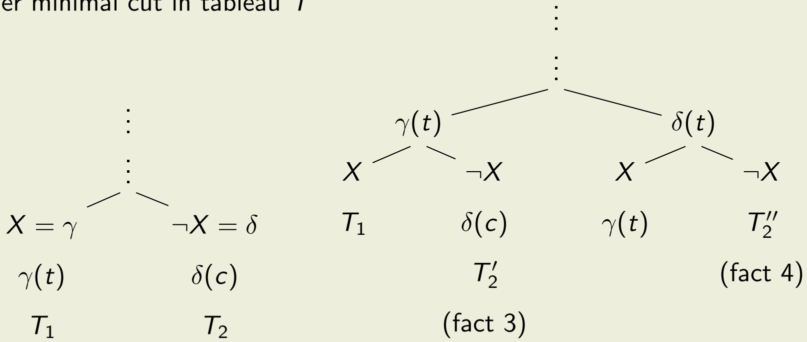


quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T

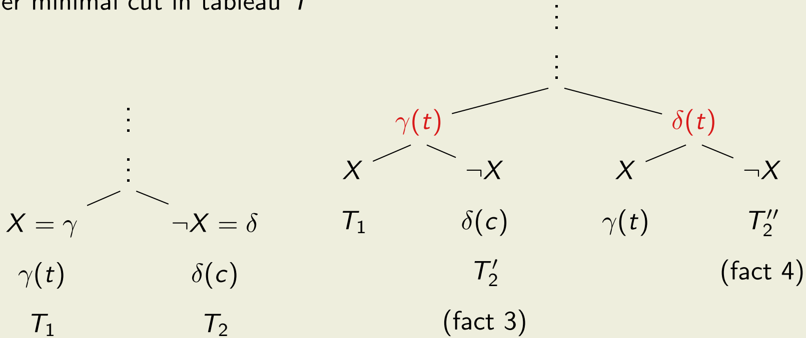


quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

Proof of Key Lemma (cont'd)

consider minimal cut in tableau T



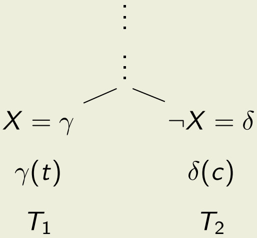
quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

rank of cut $\{\gamma(t), \delta(t)\}$ is smaller than rank of original cut $\{X, \neg X\}$

Proof of Key Lemma (cont'd)

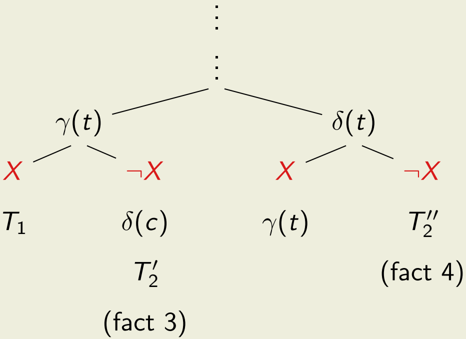
consider minimal cut in tableau T



$X = \gamma$ $\quad$ $\neg X = \delta$

$\gamma(t)$ $\quad$ $\delta(c)$

T_1 $\quad$ T_2



$\gamma(t)$ $\quad$ $\delta(t)$

X $\quad$ $\neg X$ $\quad$ X $\quad$ $\neg X$

T_1 $\quad$ $\delta(c)$ $\quad$ $\gamma(t)$ $\quad$ T_2''

T_2' $\quad$ (fact 4)

(fact 3)

quantifier case: $X = \gamma$ and $\neg X = \delta$

weight of cut is $|T_1| + |T_2| + 2$

rank of cut $\{\gamma(t), \delta(t)\}$ is smaller than rank of original cut $\{X, \neg X\}$

weight of new cuts $\{X, \neg X\}$ is $|T_1| + |T_2| + 1 < |T_1| + |T_2| + 2$

Outline

- Overview of this lecture
- Transforming Hilbert Style proof into tableau proof with cut
- Gentzen's Hauptsatz: Cut Elimination
- Craig's Interpolation Theorem
- Prenex Form
- Exercises
- Further Reading

Definition

sentence Z is interpolant for pair (S_1, S_2) of sets of sentences if all constant, function and relation symbols of Z occur in formulas of both S_1 and S_2

Definition

sentence Z is interpolant for pair (S_1, S_2) of sets of sentences if all constant, function and relation symbols of Z occur in formulas of both S_1 and S_2, and neither $S_1 \cup \{Z\}$ nor $S_2 \cup \{\neg Z\}$ is satisfiable

Definition

sentence Z is interpolant for pair (S_1, S_2) of sets of sentences if all constant, function and relation symbols of Z occur in formulas of both S_1 and S_2, and neither $S_1 \cup \{Z\}$ nor $S_2 \cup \{\neg Z\}$ is satisfiable

Definition

finite set S of sentences is Craig consistent if there exists partition (S_1, S_2) of S that lacks interpolant

Definition

sentence Z is interpolant for pair (S_1, S_2) of sets of sentences if all constant, function and relation symbols of Z occur in formulas of both S_1 and S_2, and neither $S_1 \cup \{Z\}$ nor $S_2 \cup \{\neg Z\}$ is satisfiable

Definition

finite set S of sentences is Craig consistent if there exists partition (S_1, S_2) of S that lacks interpolant

Lemma

collection of all Craig consistent sets is first-order consistency property

Definition

sentence Z is interpolant for pair (S_1, S_2) of sets of sentences if all constant, function and relation symbols of Z occur in formulas of both S_1 and S_2, and neither $S_1 \cup \{Z\}$ nor $S_2 \cup \{\neg Z\}$ is satisfiable

Definition

finite set S of sentences is Craig consistent if there exists partition (S_1, S_2) of S that lacks interpolant

Lemma

collection of all Craig consistent sets is first-order consistency property

Proof (two cases)

γ-case and δ-case $(\dots)$

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z
- $S_2 \cup \{\neg Z\}$ and $S_1 \cup \{\gamma(t), Z\}$ are not satisfiable

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z
- $S_2 \cup \{\neg Z\}$ and $S_1 \cup \{ \qquad Z\}$ are not satisfiable

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z
- $S_2 \cup \{\neg Z\}$ and $S_1 \cup \{\qquad Z\}$ are not satisfiable
- all constant, function and relation symbols of Z occur in $S_1 \cup \{\gamma(t)\}$ and S_2

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z
- $S_2 \cup \{\neg Z\}$ and $S_1 \cup \{\qquad Z\}$ are not satisfiable
- all constant, function and relation symbols of Z occur in $S_1 \cup \{\gamma(t)\}$ and S_2 and if they all occur in S_1 then Z is interpolant for (S_1, S_2)

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z
- $S_2 \cup \{\neg Z\}$ and $S_1 \cup \{\qquad Z\}$ are not satisfiable
- all constant, function and relation symbols of Z occur in $S_1 \cup \{\gamma(t)\}$ and S_2 and if they all occur in S_1 then Z is interpolant for (S_1, S_2) and thus S is not Craig consistent

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z
- $S_2 \cup \{\neg Z\}$ and $S_1 \cup \{\qquad Z\}$ are not satisfiable
- all constant, function and relation symbols of Z occur in $S_1 \cup \{\gamma(t)\}$ and S_2 and if they all occur in S_1 then Z is interpolant for (S_1, S_2) and thus S is not Craig consistent
- suppose Z contains symbol not occurring in S_1

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z
- $S_2 \cup \{\neg Z\}$ and $S_1 \cup \{\qquad Z\}$ are not satisfiable
- all constant, function and relation symbols of Z occur in $S_1 \cup \{\gamma(t)\}$ and S_2 and if they all occur in S_1 then Z is interpolant for (S_1, S_2) and thus S is not Craig consistent
- suppose Z contains symbol not occurring in S_1
- any such symbol must be constant or function symbol in t

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z
- $S_2 \cup \{\neg Z\}$ and $S_1 \cup \{\qquad Z\}$ are not satisfiable
- all constant, function and relation symbols of Z occur in $S_1 \cup \{\gamma(t)\}$ and S_2 and if they all occur in S_1 then Z is interpolant for (S_1, S_2) and thus S is not Craig consistent
- suppose Z contains symbol not occurring in S_1
- any such symbol must be constant or function symbol in t
- for simplicity suppose Z just contains one subterm $f(u_1, \ldots, u_n)$ with f occurring in t but not in S_1

Proof (γ-case)

- suppose $\gamma \in S$ but $S \cup \{\gamma(t)\}$ is not Craig consistent for some closed term t
- let (S_1, S_2) be partition of S and assume $\gamma \in S_1$ (case $\gamma \in S_2$ is similar)
- $(S_1 \cup \{\gamma(t)\}, S_2)$ is partition of $S \cup \{\gamma(t)\}$ and hence it has interpolant Z
- $S_2 \cup \{\neg Z\}$ and $S_1 \cup \{\qquad Z\}$ are not satisfiable
- all constant, function and relation symbols of Z occur in $S_1 \cup \{\gamma(t)\}$ and S_2 and if they all occur in S_1 then Z is interpolant for (S_1, S_2) and thus S is not Craig consistent
- suppose Z contains symbol not occurring in S_1
- any such symbol must be constant or function symbol in t
- for simplicity suppose Z just contains one subterm $f(u_1, \ldots, u_n)$ with f occurring in t but not in S_1
- let Z^* be obtained from Z by replacing $f(u_1, \ldots, u_n)$ with new free variable x

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2)

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x)Z^*$ occur in S_1 and in S_2

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x)Z^*$ occur in S_1 and in S_2
- $S_2 \cup \{\neg(\exists x)Z^*\}$ is unsatisfiable because $S_2 \cup \{\neg Z\}$ is unsatisfiable and $Z = Z^*\{x/f(u_1, \ldots, u_n)\} \supset (\exists x)Z^*$ is valid

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x)Z^*$ occur in S_1 and in S_2
- $S_2 \cup \{\neg(\exists x)Z^*\}$ is unsatisfiable because $S_2 \cup \{\neg Z\}$ is unsatisfiable and $Z = Z^*\{x/f(u_1, \ldots, u_n)\} \supset (\exists x)Z^*$ is valid
- suppose $S_1 \cup \{(\exists x)Z^*\}$ is satisfiable

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x)Z^*$ occur in S_1 and in S_2
- $S_2 \cup \{\neg(\exists x)Z^*\}$ is unsatisfiable because $S_2 \cup \{\neg Z\}$ is unsatisfiable and $Z = Z^*\{x/f(u_1, \ldots, u_n)\} \supset (\exists x)Z^*$ is valid
- suppose $S_1 \cup \{(\exists x)Z^*\}$ is satisfiable in model $\langle \mathbf{D}, \mathbf{I} \rangle$

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x)Z^*$ occur in S_1 and in S_2
- $S_2 \cup \{\neg(\exists x)Z^*\}$ is unsatisfiable because $S_2 \cup \{\neg Z\}$ is unsatisfiable and $Z = Z^*\{x/f(u_1, \dots, u_n)\} \supset (\exists x)Z^*$ is valid
- suppose $S_1 \cup \{(\exists x)Z^*\}$ is satisfiable in model $\langle \mathbf{D}, \mathbf{I} \rangle$

  $(Z^*)^{\mathbf{I},\mathbf{A}}$ is true for some assignment $\mathbf{A}$

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x)Z^*$ occur in S_1 and in S_2
- $S_2 \cup \{\neg(\exists x)Z^*\}$ is unsatisfiable because $S_2 \cup \{\neg Z\}$ is unsatisfiable and $Z = Z^*\{x/f(u_1,\ldots,u_n)\} \supset (\exists x)Z^*$ is valid
- suppose $S_1 \cup \{(\exists x)Z^*\}$ is satisfiable in model $\langle \mathbf{D}, \mathbf{I} \rangle$

  $(Z^*)^{\mathbf{I},\mathbf{A}}$ is true for some assignment $\mathbf{A}$

  modify interpretation $\mathbf{I}$ to $\mathbf{J}$ by changing $f^{\mathbf{I}}$ to $f^{\mathbf{J}}$ such that

$$f^{\mathbf{J}}(d_1,\ldots,d_n) = \begin{cases} x^{\mathbf{A}} & \text{if } d_i = u_i^{\mathbf{I},\mathbf{A}} \text{ for } 1 \leqslant i \leqslant n \\ f^{\mathbf{I}}(d_1,\ldots,d_n) & \text{otherwise} \end{cases}$$

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x)Z^*$ occur in S_1 and in S_2
- $S_2 \cup \{\neg(\exists x)Z^*\}$ is unsatisfiable because $S_2 \cup \{\neg Z\}$ is unsatisfiable and $Z = Z^*\{x/f(u_1, \dots, u_n)\} \supset (\exists x)Z^*$ is valid
- suppose $S_1 \cup \{(\exists x)Z^*\}$ is satisfiable in model $\langle \mathbf{D}, \mathbf{I} \rangle$

  $(Z^*)^{\mathbf{I},\mathbf{A}}$ is true for some assignment $\mathbf{A}$

  modify interpretation $\mathbf{I}$ to $\mathbf{J}$ by changing $f^{\mathbf{I}}$ to $f^{\mathbf{J}}$ such that

  $$f^{\mathbf{J}}(d_1, \dots, d_n) = \begin{cases} x^{\mathbf{A}} & \text{if } d_i = u_i^{\mathbf{I},\mathbf{A}} \text{ for } 1 \leqslant i \leqslant n \\ f^{\mathbf{I}}(d_1, \dots, d_n) & \text{otherwise} \end{cases}$$

  all sentences in S_1 are true in $\langle \mathbf{D}, \mathbf{J} \rangle$ because f does not occur in S_1

Proof (γ-case, cont'd)

$(\exists x) Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x) Z^*$ occur in S_1 and in S_2
- $S_2 \cup \{\neg(\exists x) Z^*\}$ is unsatisfiable because $S_2 \cup \{\neg Z\}$ is unsatisfiable and $Z = Z^*\{x/f(u_1, \dots, u_n)\} \supset (\exists x) Z^*$ is valid
- suppose $S_1 \cup \{(\exists x) Z^*\}$ is satisfiable in model $\langle \mathbf{D}, \mathbf{I} \rangle$

  $(Z^*)^{\mathbf{I},\mathbf{A}}$ is true for some assignment $\mathbf{A}$

  modify interpretation $\mathbf{I}$ to $\mathbf{J}$ by changing $f^{\mathbf{I}}$ to $f^{\mathbf{J}}$ such that

$$f^{\mathbf{J}}(d_1, \dots, d_n) = \begin{cases} x^{\mathbf{A}} & \text{if } d_i = u_i^{\mathbf{I},\mathbf{A}} \text{ for } 1 \leqslant i \leqslant n \\ f^{\mathbf{I}}(d_1, \dots, d_n) & \text{otherwise} \end{cases}$$

  all sentences in S_1 are true in $\langle \mathbf{D}, \mathbf{J} \rangle$ because f does not occur in S_1

  $Z^{\mathbf{J},\mathbf{A}} = [Z^*\{x/f(u_1, \dots, u_n)\}]^{\mathbf{J},\mathbf{A}} = (Z^*)^{\mathbf{J},\mathbf{A}} = (Z^*)^{\mathbf{I},\mathbf{A}} = \mathrm{t}$

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x)Z^*$ occur in S_1 and in S_2
- $S_2 \cup \{\neg(\exists x)Z^*\}$ is unsatisfiable because $S_2 \cup \{\neg Z\}$ is unsatisfiable and $Z = Z^*\{x/f(u_1,\dots,u_n)\} \supset (\exists x)Z^*$ is valid
- suppose $S_1 \cup \{(\exists x)Z^*\}$ is satisfiable in model $\langle \mathbf{D}, \mathbf{I} \rangle$

  $(Z^*)^{\mathbf{I},\mathbf{A}}$ is true for some assignment $\mathbf{A}$

  modify interpretation $\mathbf{I}$ to $\mathbf{J}$ by changing $f^{\mathbf{I}}$ to $f^{\mathbf{J}}$ such that

$$f^{\mathbf{J}}(d_1,\dots,d_n) = \begin{cases} x^{\mathbf{A}} & \text{if } d_i = u_i^{\mathbf{I},\mathbf{A}} \text{ for } 1 \leqslant i \leqslant n \\ f^{\mathbf{I}}(d_1,\dots,d_n) & \text{otherwise} \end{cases}$$

  all sentences in S_1 are true in $\langle \mathbf{D}, \mathbf{J} \rangle$ because f does not occur in S_1

  $Z^{\mathbf{J},\mathbf{A}} = [Z^*\{x/f(u_1,\dots,u_n)\}]^{\mathbf{J},\mathbf{A}} = (Z^*)^{\mathbf{J},\mathbf{A}} = (Z^*)^{\mathbf{I},\mathbf{A}} = \mathrm{t}$

  $S_1 \cup \{Z\}$ is satisfiable

Proof (γ-case, cont'd)

$(\exists x)Z^*$ is interpolant for (S_1, S_2):

- all constant, function and relation symbols of $(\exists x)Z^*$ occur in S_1 and in S_2
- $S_2 \cup \{\neg(\exists x)Z^*\}$ is unsatisfiable because $S_2 \cup \{\neg Z\}$ is unsatisfiable and $Z = Z^*\{x/f(u_1, \ldots, u_n)\} \supset (\exists x)Z^*$ is valid
- suppose $S_1 \cup \{(\exists x)Z^*\}$ is satisfiable in model $\langle \mathbf{D}, \mathbf{I} \rangle$

  $(Z^*)^{\mathbf{I},\mathbf{A}}$ is true for some assignment $\mathbf{A}$

  modify interpretation $\mathbf{I}$ to $\mathbf{J}$ by changing $f^{\mathbf{I}}$ to $f^{\mathbf{J}}$ such that

  $$f^{\mathbf{J}}(d_1, \ldots, d_n) = \begin{cases} x^{\mathbf{A}} & \text{if } d_i = u_i^{\mathbf{I},\mathbf{A}} \text{ for } 1 \leqslant i \leqslant n \\ f^{\mathbf{I}}(d_1, \ldots, d_n) & \text{otherwise} \end{cases}$$

  all sentences in S_1 are true in $\langle \mathbf{D}, \mathbf{J} \rangle$ because f does not occur in S_1

  $Z^{\mathbf{J},\mathbf{A}} = [Z^*\{x/f(u_1, \ldots, u_n)\}]^{\mathbf{J},\mathbf{A}} = (Z^*)^{\mathbf{J},\mathbf{A}} = (Z^*)^{\mathbf{I},\mathbf{A}} = \mathrm{t}$

  $S_1 \cup \{Z\}$ is satisfiable ↯

Proof (δ-case)

- suppose $\delta \in S$ but, for each parameter p, $S \cup \{\delta(p)\}$ is not Craig consistent

Proof (δ-case)

- suppose $\delta \in S$ but, for each parameter p, $S \cup \{\delta(p)\}$ is not Craig consistent
- let (S_1, S_2) be partition of S and assume $\delta \in S_1$

Proof (δ-case)

- suppose $\delta \in S$ but, for each parameter p, $S \cup \{\delta(p)\}$ is not Craig consistent
- let (S_1, S_2) be partition of S and assume $\delta \in S_1$
- let p be parameter that does not occur in S

Proof (δ-case)

- suppose $\delta \in S$ but, for each parameter p, $S \cup \{\delta(p)\}$ is not Craig consistent
- let (S_1, S_2) be partition of S and assume $\delta \in S_1$
- let p be parameter that does not occur in S
- $(S_1 \cup \{\delta(p)\}, S_2)$ is partition of $S \cup \{\delta(p)\}$ and hence it has interpolant Z

Proof (δ-case)

- suppose $\delta \in S$ but, for each parameter p, $S \cup \{\delta(p)\}$ is not Craig consistent
- let (S_1, S_2) be partition of S and assume $\delta \in S_1$
- let p be parameter that does not occur in S
- $(S_1 \cup \{\delta(p)\}, S_2)$ is partition of $S \cup \{\delta(p)\}$ and hence it has interpolant Z
- Z is interpolant for (S_1, S_2)

Proof (δ-case)

- suppose $\delta \in S$ but, for each parameter p, $S \cup \{\delta(p)\}$ is not Craig consistent
- let (S_1, S_2) be partition of S and assume $\delta \in S_1$
- let p be parameter that does not occur in S
- $(S_1 \cup \{\delta(p)\}, S_2)$ is partition of $S \cup \{\delta(p)\}$ and hence it has interpolant Z
- Z is interpolant for (S_1, S_2):
    - all constant, function and relation symbols of Z occur in S_1 and in S_2

Proof (δ-case)

- suppose $\delta \in S$ but, for each parameter p, $S \cup \{\delta(p)\}$ is not Craig consistent
- let (S_1, S_2) be partition of S and assume $\delta \in S_1$
- let p be parameter that does not occur in S
- $(S_1 \cup \{\delta(p)\}, S_2)$ is partition of $S \cup \{\delta(p)\}$ and hence it has interpolant Z
- Z is interpolant for (S_1, S_2):
  - all constant, function and relation symbols of Z occur in S_1 and in S_2
  - $S_2 \cup \{\neg Z\}$ is unsatisfiable

Proof (δ-case)

- suppose $\delta \in S$ but, for each parameter p, $S \cup \{\delta(p)\}$ is not Craig consistent
- let (S_1, S_2) be partition of S and assume $\delta \in S_1$
- let p be parameter that does not occur in S
- $(S_1 \cup \{\delta(p)\}, S_2)$ is partition of $S \cup \{\delta(p)\}$ and hence it has interpolant Z
- Z is interpolant for (S_1, S_2):
    - all constant, function and relation symbols of Z occur in S_1 and in S_2
    - $S_2 \cup \{\neg Z\}$ is unsatisfiable
    - $S_1 \cup \{\delta(p), Z\}$ is unsatisfiable

Proof (δ-case)

- suppose $\delta \in S$ but, for each parameter p, $S \cup \{\delta(p)\}$ is not Craig consistent
- let (S_1, S_2) be partition of S and assume $\delta \in S_1$
- let p be parameter that does not occur in S
- $(S_1 \cup \{\delta(p)\}, S_2)$ is partition of $S \cup \{\delta(p)\}$ and hence it has interpolant Z
- Z is interpolant for (S_1, S_2):
  - all constant, function and relation symbols of Z occur in S_1 and in S_2
  - $S_2 \cup \{\neg Z\}$ is unsatisfiable
  - $S_1 \cup \{\delta(p), Z\}$ is unsatisfiable and hence $S_1 \cup \{Z\}$ is unsatisfiable by reasoning like in γ-case

Definition

sentence Z is interpolant for sentence $X \supset Y$ if all constant, function and relation symbols of Z are common to X and Y, and both $X \supset Z$ and $Z \supset Y$ are valid

Definition

sentence Z is interpolant for sentence $X \supset Y$ if all constant, function and relation symbols of Z are common to X and Y, and both $X \supset Z$ and $Z \supset Y$ are valid

Theorem (First-Order Craig Interpolation)

every valid sentence $X \supset Y$ has interpolant

Definition

sentence Z is interpolant for sentence $X \supset Y$ if all constant, function and relation symbols of Z are common to X and Y, and both $X \supset Z$ and $Z \supset Y$ are valid

Theorem (First-Order Craig Interpolation)

every valid sentence $X \supset Y$ *has interpolant*

Proof

- suppose $X \supset Y$ lacks interpolant

Definition

sentence Z is interpolant for sentence $X \supset Y$ if all constant, function and relation symbols of Z are common to X and Y, and both $X \supset Z$ and $Z \supset Y$ are valid

Theorem (First-Order Craig Interpolation)

every valid sentence $X \supset Y$ has interpolant

Proof

- suppose $X \supset Y$ lacks interpolant
- $S = \{X, \neg Y\}$ with partition $S_1 = \{\neg Y\}$ and $S_2 = \{X\}$

Definition

sentence Z is interpolant for sentence $X \supset Y$ if all constant, function and relation symbols of Z are common to X and Y, and both $X \supset Z$ and $Z \supset Y$ are valid

Theorem (First-Order Craig Interpolation)

every valid sentence $X \supset Y$ *has interpolant*

Proof

- suppose $X \supset Y$ lacks interpolant
- $S = \{X, \neg Y\}$ with partition $S_1 = \{\neg Y\}$ and $S_2 = \{X\}$
- if (S_1, S_2) has interpolant Z then Z is interpolant for $X \supset Y$

Definition

sentence Z is interpolant for sentence $X \supset Y$ if all constant, function and relation symbols of Z are common to X and Y, and both $X \supset Z$ and $Z \supset Y$ are valid

Theorem (First-Order Craig Interpolation)

every valid sentence $X \supset Y$ *has interpolant*

Proof

- suppose $X \supset Y$ lacks interpolant
- $S = \{X, \neg Y\}$ with partition $S_1 = \{\neg Y\}$ and $S_2 = \{X\}$
- if (S_1, S_2) has interpolant Z then Z is interpolant for $X \supset Y$
- S is Craig consistent

Definition

sentence Z is interpolant for sentence $X \supset Y$ if all constant, function and relation symbols of Z are common to X and Y, and both $X \supset Z$ and $Z \supset Y$ are valid

Theorem (First-Order Craig Interpolation)

every valid sentence $X \supset Y$ *has interpolant*

Proof

- suppose $X \supset Y$ lacks interpolant
- $S = \{X, \neg Y\}$ with partition $S_1 = \{\neg Y\}$ and $S_2 = \{X\}$
- if (S_1, S_2) has interpolant Z then Z is interpolant for $X \supset Y$
- S is Craig consistent and hence S is satisfiable by Model Existence Theorem

Definition

sentence Z is interpolant for sentence $X \supset Y$ if all constant, function and relation symbols of Z are common to X and Y, and both $X \supset Z$ and $Z \supset Y$ are valid

Theorem (First-Order Craig Interpolation)

every valid sentence $X \supset Y$ has interpolant

Proof

- suppose $X \supset Y$ lacks interpolant
- $S = \{X, \neg Y\}$ with partition $S_1 = \{\neg Y\}$ and $S_2 = \{X\}$
- if (S_1, S_2) has interpolant Z then Z is interpolant for $X \supset Y$
- S is Craig consistent and hence S is satisfiable by Model Existence Theorem
- $X \supset Y$ is not valid

Definition

biased sentence is expression $L(Z)$ or $R(Z)$ where Z is sentence

Definition

biased sentence is expression $L(Z)$ or $R(Z)$ where Z is sentence

tableau proof of $X \supset Y$

$$\begin{array}{c} \neg(X \supset Y) \\ X \\ \neg Y \\ T \end{array}$$

Definition

biased sentence is expression $L(Z)$ or $R(Z)$ where Z is sentence

tableau proof of $X \supset Y$

$$L(X)$$
$$R(\neg Y)$$
$$T'$$

can be transformed into closed biased tableau for $\{L(X), R(\neg Y)\}$

Definition

biased sentence is expression $L(Z)$ or $R(Z)$ where Z is sentence

tableau proof of $X \supset Y$

$$\begin{array}{c} L(X) \\ R(\neg Y) \\ T' \end{array} \qquad \frac{L(\alpha)}{\begin{array}{c} L(\alpha_1) \\ L(\alpha_2) \end{array}} \quad \frac{R(\alpha)}{\begin{array}{c} R(\alpha_1) \\ R(\alpha_2) \end{array}} \quad \frac{L(\beta)}{L(\beta_1) \mid L(\beta_2)} \quad \frac{R(\beta)}{R(\beta_1) \mid R(\beta_2)} \quad \cdots$$

can be transformed into closed biased tableau for $\{L(X), R(\neg Y)\}$

Definition

sentence Z is interpolant for finite set $\{L(A_1), \ldots, L(A_n), R(B_1), \ldots, R(B_k)\}$ provided Z is interpolant for sentence $(A_1 \wedge \cdots \wedge A_n) \supset (\neg B_1 \vee \cdots \vee \neg B_k)$

Definition

sentence Z is interpolant for finite set $\{L(A_1), \ldots, L(A_n), R(B_1), \ldots, R(B_k)\}$ provided Z is interpolant for sentence $(A_1 \wedge \cdots \wedge A_n) \supset (\neg B_1 \vee \cdots \vee \neg B_k)$

Notation

$S \xrightarrow{\text{int}} Z$ denotes that Z is interpolant for finite set S of biased sentences

Definition

sentence Z is interpolant for finite set $\{L(A_1), \dots, L(A_n), R(B_1), \dots, R(B_k)\}$ provided Z is interpolant for sentence $(A_1 \wedge \cdots \wedge A_n) \supset (\neg B_1 \vee \cdots \vee \neg B_k)$

Notation

$S \xrightarrow{\text{int}} Z$ denotes that Z is interpolant for finite set S of biased sentences

Calculation Rules for Interpolants

$$S \cup \{L(\bot)\} \xrightarrow{\text{int}} \bot$$

Definition

sentence Z is interpolant for finite set $\{L(A_1), \dots, L(A_n), R(B_1), \dots, R(B_k)\}$ provided Z is interpolant for sentence $(A_1 \wedge \cdots \wedge A_n) \supset (\neg B_1 \vee \cdots \vee \neg B_k)$

Notation

$S \xrightarrow{\text{int}} Z$ denotes that Z is interpolant for finite set S of biased sentences

Calculation Rules for Interpolants

$$S \cup \{L(\bot)\} \xrightarrow{\text{int}} \bot$$

$$S \cup \{R(\bot)\} \xrightarrow{\text{int}} \top$$

Definition

sentence Z is interpolant for finite set $\{L(A_1), \dots, L(A_n), R(B_1), \dots, R(B_k)\}$ provided Z is interpolant for sentence $(A_1 \wedge \cdots \wedge A_n) \supset (\neg B_1 \vee \cdots \vee \neg B_k)$

Notation

$S \xrightarrow{\text{int}} Z$ denotes that Z is interpolant for finite set S of biased sentences

Calculation Rules for Interpolants

$$S \cup \{L(\bot)\} \xrightarrow{\text{int}} \bot$$

$$S \cup \{R(\bot)\} \xrightarrow{\text{int}} \top$$

$$S \cup \{L(A), L(\neg A)\} \xrightarrow{\text{int}} \bot$$

Definition

sentence Z is interpolant for finite set $\{L(A_1), \dots, L(A_n), R(B_1), \dots, R(B_k)\}$ provided Z is interpolant for sentence $(A_1 \wedge \cdots \wedge A_n) \supset (\neg B_1 \vee \cdots \vee \neg B_k)$

Notation

$S \xrightarrow{\text{int}} Z$ denotes that Z is interpolant for finite set S of biased sentences

Calculation Rules for Interpolants

$$S \cup \{L(\bot)\} \xrightarrow{\text{int}} \bot$$

$$S \cup \{R(\bot)\} \xrightarrow{\text{int}} \top$$

$$S \cup \{L(A), L(\neg A)\} \xrightarrow{\text{int}} \bot \qquad S \cup \{R(A), R(\neg A)\} \xrightarrow{\text{int}} \top$$

Definition

sentence Z is interpolant for finite set $\{L(A_1), \dots, L(A_n), R(B_1), \dots, R(B_k)\}$ provided Z is interpolant for sentence $(A_1 \wedge \cdots \wedge A_n) \supset (\neg B_1 \vee \cdots \vee \neg B_k)$

Notation

$S \xrightarrow{\text{int}} Z$ denotes that Z is interpolant for finite set S of biased sentences

Calculation Rules for Interpolants

$$S \cup \{L(\bot)\} \xrightarrow{\text{int}} \bot$$

$$S \cup \{R(\bot)\} \xrightarrow{\text{int}} \top$$

$$S \cup \{L(A), L(\neg A)\} \xrightarrow{\text{int}} \bot \qquad S \cup \{R(A), R(\neg A)\} \xrightarrow{\text{int}} \top$$

$$S \cup \{L(A), R(\neg A)\} \xrightarrow{\text{int}} A$$

Definition

sentence Z is interpolant for finite set $\{L(A_1), \dots, L(A_n), R(B_1), \dots, R(B_k)\}$ provided Z is interpolant for sentence $(A_1 \wedge \cdots \wedge A_n) \supset (\neg B_1 \vee \cdots \vee \neg B_k)$

Notation

$S \xrightarrow{\text{int}} Z$ denotes that Z is interpolant for finite set S of biased sentences

Calculation Rules for Interpolants

$$S \cup \{L(\bot)\} \xrightarrow{\text{int}} \bot$$

$$S \cup \{R(\bot)\} \xrightarrow{\text{int}} \top$$

$$S \cup \{L(A), L(\neg A)\} \xrightarrow{\text{int}} \bot \qquad S \cup \{R(A), R(\neg A)\} \xrightarrow{\text{int}} \top$$

$$S \cup \{L(A), R(\neg A)\} \xrightarrow{\text{int}} A \qquad S \cup \{R(A), L(\neg A)\} \xrightarrow{\text{int}} \neg A$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\top)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\bot)\} \xrightarrow{\text{int}} A}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\top)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\bot)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{L(\bot)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\top)\} \xrightarrow{\text{int}} A}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\top)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\bot)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{L(\bot)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\top)\} \xrightarrow{\text{int}} A}$$

$$\frac{S \cup \{R(\top)\} \xrightarrow{\text{int}} A}{S \cup \{R(\neg\bot)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\bot)\} \xrightarrow{\text{int}} A}{S \cup \{R(\neg\top)\} \xrightarrow{\text{int}} A}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\top)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\bot)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{L(\bot)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\top)\} \xrightarrow{\text{int}} A}$$

$$\frac{S \cup \{R(\top)\} \xrightarrow{\text{int}} A}{S \cup \{R(\neg\bot)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\bot)\} \xrightarrow{\text{int}} A}{S \cup \{R(\neg\top)\} \xrightarrow{\text{int}} A}$$

$$\frac{S \cup \{L(Z)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\neg Z)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(Z)\} \xrightarrow{\text{int}} A}{S \cup \{R(\neg\neg Z)\} \xrightarrow{\text{int}} A}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\top)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\bot)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{L(\bot)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\top)\} \xrightarrow{\text{int}} A}$$

$$\frac{S \cup \{R(\top)\} \xrightarrow{\text{int}} A}{S \cup \{R(\neg\bot)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\bot)\} \xrightarrow{\text{int}} A}{S \cup \{R(\neg\top)\} \xrightarrow{\text{int}} A}$$

$$\frac{S \cup \{L(Z)\} \xrightarrow{\text{int}} A}{S \cup \{L(\neg\neg Z)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(Z)\} \xrightarrow{\text{int}} A}{S \cup \{R(\neg\neg Z)\} \xrightarrow{\text{int}} A}$$

$$\frac{S \cup \{L(\alpha_1), L(\alpha_2)\} \xrightarrow{\text{int}} A}{S \cup \{L(\alpha)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\alpha_1), R(\alpha_2)\} \xrightarrow{\text{int}} A}{S \cup \{R(\alpha)\} \xrightarrow{\text{int}} A}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Verification

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Verification

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- A is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_1)$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Verification

suppose $S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$

- A is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_1)$
- all relation, function, and constant symbols of A appear in both $X_1 \wedge \cdots \wedge X_n$ and $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_1$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Verification

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- A is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1)$
- all relation, function, and constant symbols of A appear in both $X_1 \wedge \cdots \wedge X_n$ and $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1$ and hence also in $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Verification

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- A is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1)$
- all relation, function, and constant symbols of A appear in both $X_1 \wedge \cdots \wedge X_n$ and $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1$ and hence also in $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta$
- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1)$ are valid

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Verification (cont'd)

suppose $S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$

- B is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_2)$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Verification (cont'd)

suppose $S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$

- B is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_2)$
- all relation, function and constant symbols of B appear in both $X_1 \wedge \cdots \wedge X_n$ and $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_2$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Verification (cont'd)

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- B is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_2)$
- all relation, function and constant symbols of B appear in both $X_1 \wedge \cdots \wedge X_n$ and $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_2$ and hence also in $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{L(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{L(\beta)\} \xrightarrow{\text{int}} A \vee B}$$

$$\frac{S \cup \{R(\beta_1)\} \xrightarrow{\text{int}} A \quad S \cup \{R(\beta_2)\} \xrightarrow{\text{int}} B}{S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B}$$

Verification (cont'd)

suppose $S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$

- B is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2)$
- all relation, function and constant symbols of B appear in both $X_1 \wedge \cdots \wedge X_n$ and $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2$ and hence also in $\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta$
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2)$ are valid

Verification (cont'd)

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2)$ are valid

Verification (cont'd)

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_1)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_2)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset A \wedge B$ is valid

Verification (cont'd)

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_1)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_2)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset A \wedge B$ is valid
- $A \wedge B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta)$ is valid

Verification (cont'd)

suppose $S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$

- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset A \wedge B$ is valid
- $A \wedge B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta)$ is valid:

$$A \wedge B \ \ \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1) \wedge (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2)$$

Verification (cont'd)

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset A \wedge B$ is valid
- $A \wedge B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta)$ is valid:

$$\begin{aligned} A \wedge B \; &\supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1) \wedge (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2) \\ &\equiv \; (\neg Y_1 \vee \cdots \vee \neg Y_k \vee (\neg \beta_1 \wedge \neg \beta_2)) \end{aligned}$$

Verification (cont'd)

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_1)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_2)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset A \wedge B$ is valid
- $A \wedge B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta)$ is valid:

$$\begin{aligned} A \wedge B \ &\supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_1) \wedge (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\beta_2) \\ &\equiv \ (\neg Y_1 \vee \cdots \vee \neg Y_k \vee (\neg\beta_1 \wedge \neg\beta_2)) \\ &\equiv \ (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg(\beta_1 \vee \beta_2)) \end{aligned}$$

Verification (cont'd)

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset A \wedge B$ is valid
- $A \wedge B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta)$ is valid:

$$\begin{aligned} A \wedge B \ &\supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1) \wedge (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2) \\ &\equiv \ (\neg Y_1 \vee \cdots \vee \neg Y_k \vee (\neg \beta_1 \wedge \neg \beta_2)) \\ &\equiv \ (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg(\beta_1 \vee \beta_2)) \\ &\equiv \ (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta) \end{aligned}$$

Verification (cont'd)

suppose $S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$

- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset A \wedge B$ is valid
- $A \wedge B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta)$ is valid:

$$\begin{aligned} A \wedge B \ &\supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1) \wedge (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2) \\ &\equiv \ (\neg Y_1 \vee \cdots \vee \neg Y_k \vee (\neg \beta_1 \wedge \neg \beta_2)) \\ &\equiv \ (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg(\beta_1 \vee \beta_2)) \\ &\equiv \ (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta) \end{aligned}$$

- $A \wedge B$ is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta)$

Verification (cont'd)

suppose $S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$

- $X_1 \wedge \cdots \wedge X_n \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset B$ and $B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2)$ are valid
- $X_1 \wedge \cdots \wedge X_n \supset A \wedge B$ is valid
- $A \wedge B \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta)$ is valid:

$$\begin{aligned} A \wedge B &\supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_1) \wedge (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta_2) \\ &\equiv (\neg Y_1 \vee \cdots \vee \neg Y_k \vee (\neg \beta_1 \wedge \neg \beta_2)) \\ &\equiv (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg(\beta_1 \vee \beta_2)) \\ &\equiv (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta) \end{aligned}$$

- $A \wedge B$ is interpolant for $(X_1 \wedge \cdots \wedge X_n) \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg \beta)$
- $S \cup \{R(\beta)\} \xrightarrow{\text{int}} A \wedge B$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$

$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$

$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$R((A \vee E) \supset \neg(\neg B \supset C))$$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$

$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

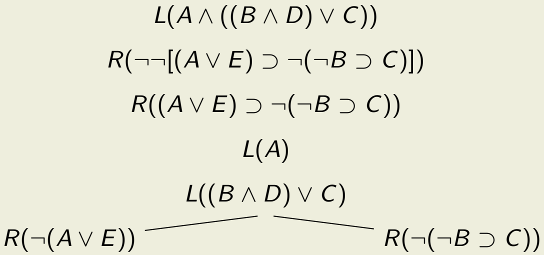
$$R((A \vee E) \supset \neg(\neg B \supset C))$$

$$L(A)$$

$$L((B \wedge D) \vee C)$$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$L(A \wedge ((B \wedge D) \vee C))$

$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$

$R((A \vee E) \supset \neg(\neg B \supset C))$

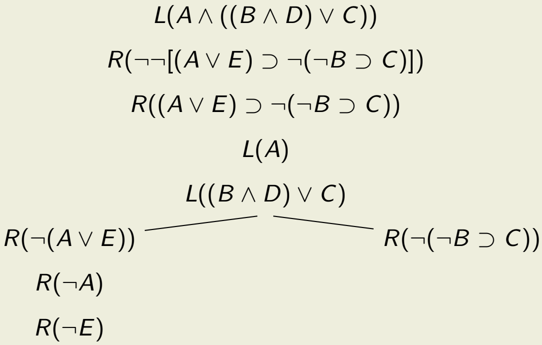
$L(A)$

$L((B \wedge D) \vee C)$

$R(\neg(A \vee E))$ $R(\neg(\neg B \supset C))$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$R((A \vee E) \supset \neg(\neg B \supset C))$$

$$L(A)$$

$$L((B \wedge D) \vee C)$$

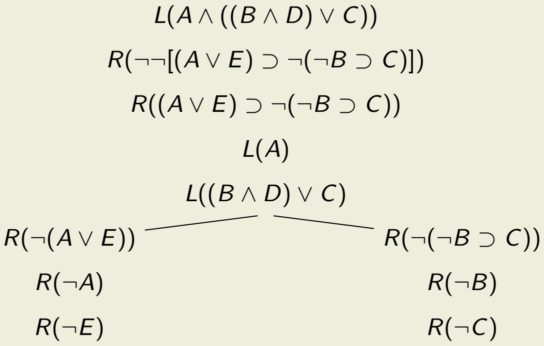
$$R(\neg(A \vee E)) \qquad R(\neg(\neg B \supset C))$$

$$R(\neg A)$$

$$R(\neg E)$$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$R((A \vee E) \supset \neg(\neg B \supset C))$$

$$L(A)$$

$$L((B \wedge D) \vee C)$$

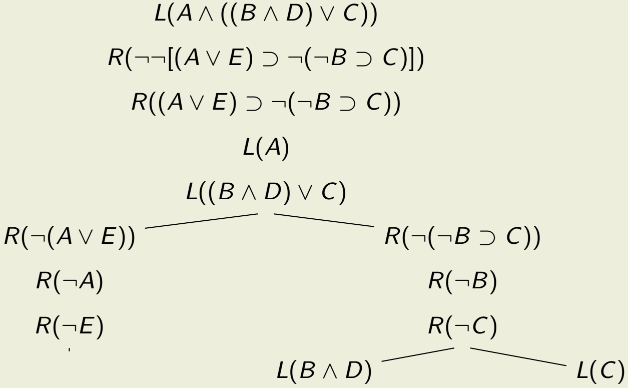
$R(\neg(A \vee E))$ | $R(\neg(\neg B \supset C))$

$R(\neg A)$ | $R(\neg B)$

$R(\neg E)$ | $R(\neg C)$

Example

interpolant for tautology $[A \land ((B \land D) \lor C] \supset \neg[(A \lor E) \supset \neg(\neg B \supset C)]$



$L(A \land ((B \land D) \lor C))$

$R(\neg\neg[(A \lor E) \supset \neg(\neg B \supset C)])$

$R((A \lor E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \land D) \lor C)$

$R(\neg(A \lor E))$ $R(\neg(\neg B \supset C))$

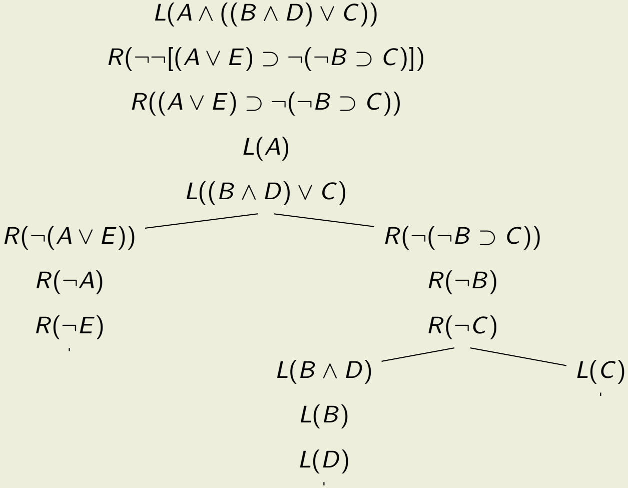
$R(\neg A)$ $R(\neg B)$

$R(\neg E)$ $R(\neg C)$

$L(B \land D)$ $L(C)$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$L(A \wedge ((B \wedge D) \vee C))$

$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$

$R((A \vee E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \wedge D) \vee C)$

$R(\neg(A \vee E))$

$R(\neg A)$

$R(\neg E)$

$R(\neg(\neg B \supset C))$

$R(\neg B)$

$R(\neg C)$

$L(B \wedge D)$

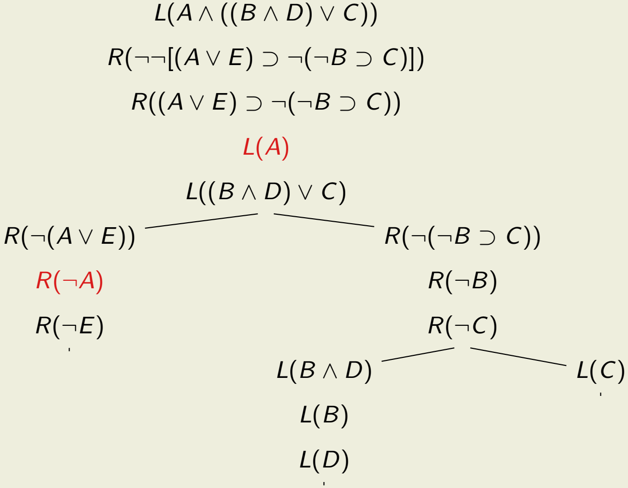
$L(B)$

$L(D)$

$L(C)$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$L(A \wedge ((B \wedge D) \vee C))$

$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$

$R((A \vee E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \wedge D) \vee C)$

$R(\neg(A \vee E))$

$R(\neg A)$

$R(\neg E)$

$R(\neg(\neg B \supset C))$

$R(\neg B)$

$R(\neg C)$

$L(B \wedge D)$

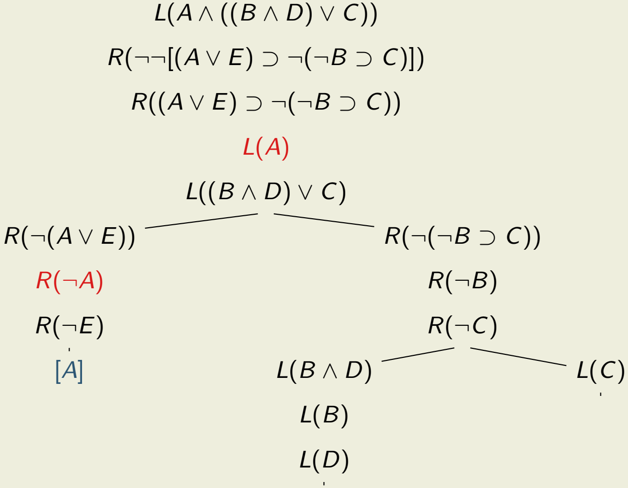
$L(B)$

$L(D)$

$L(C)$

Example

interpolant for tautology $[A \land ((B \land D) \lor C] \supset \neg[(A \lor E) \supset \neg(\neg B \supset C)]$



$L(A \land ((B \land D) \lor C))$

$R(\neg\neg[(A \lor E) \supset \neg(\neg B \supset C)])$

$R((A \lor E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \land D) \lor C)$

$R(\neg(A \lor E))$

$R(\neg A)$

$R(\neg E)$

$[A]$

$R(\neg(\neg B \supset C))$

$R(\neg B)$

$R(\neg C)$

$L(B \land D)$

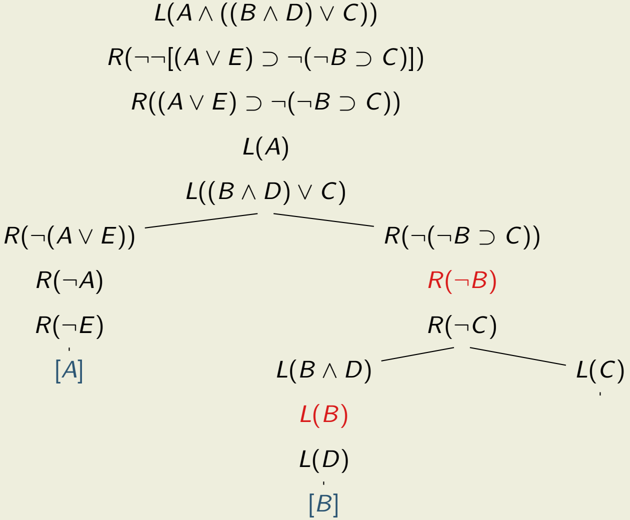
$L(B)$

$L(D)$

$L(C)$

Example

interpolant for tautology $[A \land ((B \land D) \lor C] \supset \neg[(A \lor E) \supset \neg(\neg B \supset C)]$



$L(A \land ((B \land D) \lor C))$

$R(\neg\neg[(A \lor E) \supset \neg(\neg B \supset C)])$

$R((A \lor E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \land D) \lor C)$

$R(\neg(A \lor E))$

$R(\neg A)$

$R(\neg E)$

$[A]$

$R(\neg(\neg B \supset C))$

$R(\neg B)$

$R(\neg C)$

$L(B \land D)$

$L(B)$

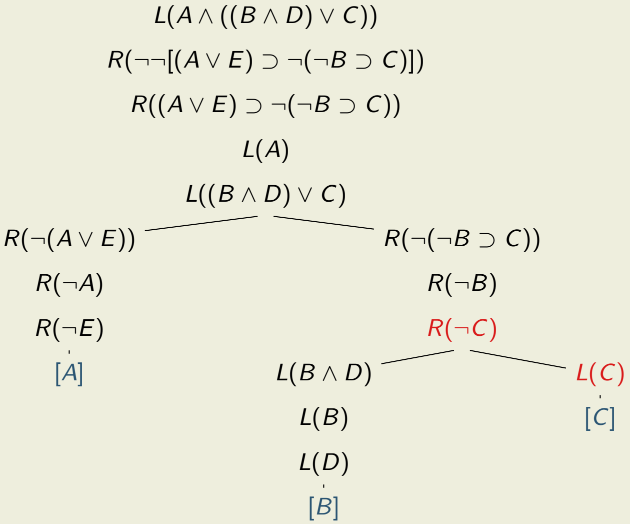
$L(D)$

$[B]$

$L(C)$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$L(A \wedge ((B \wedge D) \vee C))$

$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$

$R((A \vee E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \wedge D) \vee C)$

$R(\neg(A \vee E))$

$R(\neg A)$

$R(\neg E)$

$[A]$

$R(\neg(\neg B \supset C))$

$R(\neg B)$

$R(\neg C)$

$L(B \wedge D)$

$L(B)$

$L(D)$

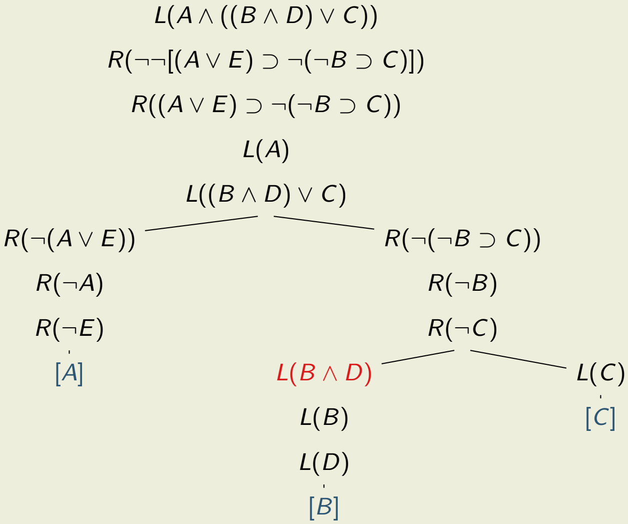
$[B]$

$L(C)$

$[C]$

Example

interpolant for tautology $[A \land ((B \land D) \lor C] \supset \neg[(A \lor E) \supset \neg(\neg B \supset C)]$



$L(A \land ((B \land D) \lor C))$

$R(\neg\neg[(A \lor E) \supset \neg(\neg B \supset C)])$

$R((A \lor E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \land D) \lor C)$

$R(\neg(A \lor E))$

$R(\neg A)$

$R(\neg E)$

$[A]$

$R(\neg(\neg B \supset C))$

$R(\neg B)$

$R(\neg C)$

$L(B \land D)$

$L(B)$

$L(D)$

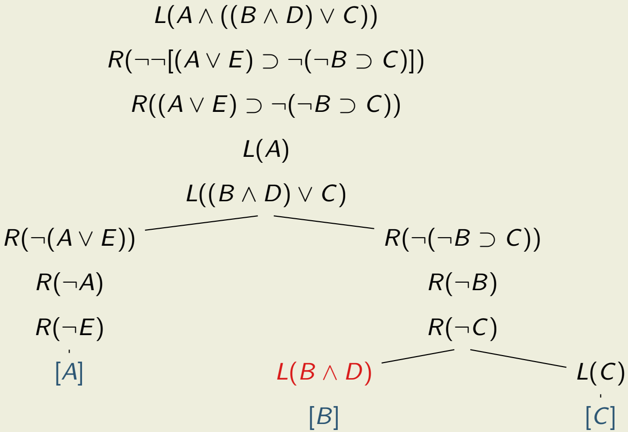
$[B]$

$L(C)$

$[C]$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$L(A \wedge ((B \wedge D) \vee C))$

$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$

$R((A \vee E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \wedge D) \vee C)$

$R(\neg(A \vee E))$ $R(\neg(\neg B \supset C))$

$R(\neg A)$ $R(\neg B)$

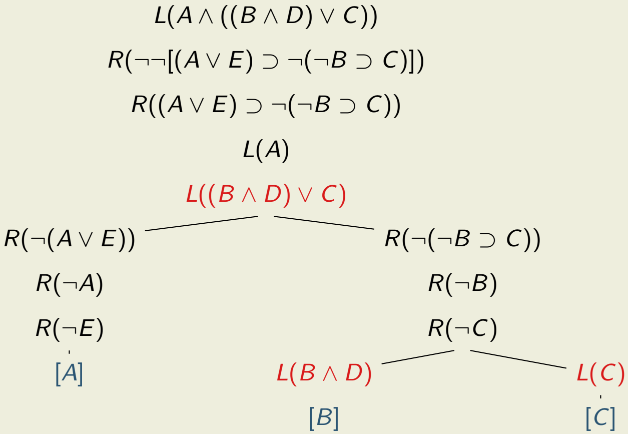
$R(\neg E)$ $R(\neg C)$

$[A]$ $L(B \wedge D)$ $L(C)$

$[B]$ $[C]$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$L(A \wedge ((B \wedge D) \vee C))$

$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$

$R((A \vee E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \wedge D) \vee C)$

$R(\neg(A \vee E))$ | $R(\neg(\neg B \supset C))$

$R(\neg A)$ | $R(\neg B)$

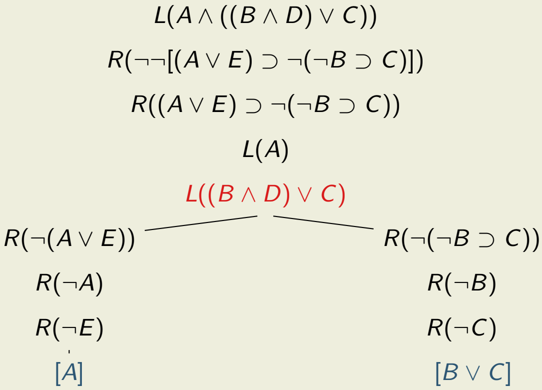
$R(\neg E)$ | $R(\neg C)$

$[A]$ | $L(B \wedge D)$ | $L(C)$

$[B]$ | $[C]$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$L(A \wedge ((B \wedge D) \vee C))$

$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$

$R((A \vee E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \wedge D) \vee C)$

$R(\neg(A \vee E))$

$R(\neg A)$

$R(\neg E)$

$[A]$

$R(\neg(\neg B \supset C))$

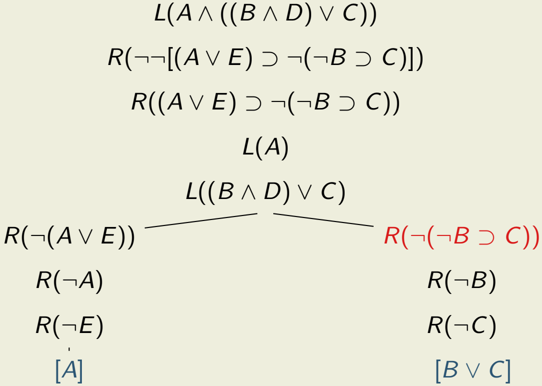
$R(\neg B)$

$R(\neg C)$

$[B \vee C]$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$R((A \vee E) \supset \neg(\neg B \supset C))$$

$$L(A)$$

$$L((B \wedge D) \vee C)$$

$R(\neg(A \vee E))$

$R(\neg A)$

$R(\neg E)$

$[A]$

$R(\neg(\neg B \supset C))$

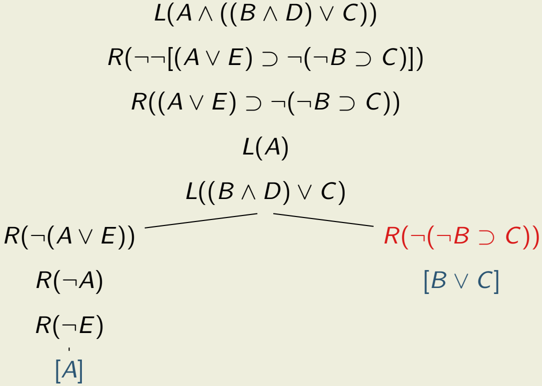
$R(\neg B)$

$R(\neg C)$

$[B \vee C]$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$L(A \wedge ((B \wedge D) \vee C))$

$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$

$R((A \vee E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \wedge D) \vee C)$

$R(\neg(A \vee E))$ $\quad$ $R(\neg(\neg B \supset C))$

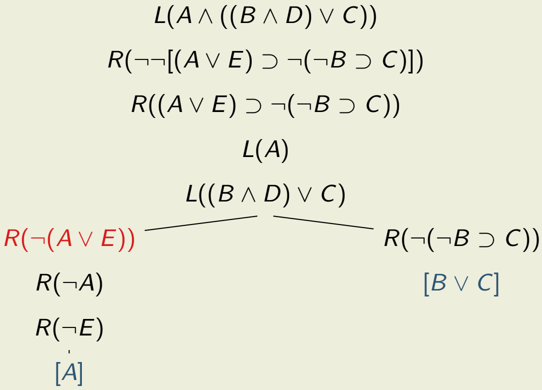
$R(\neg A)$ $\quad$ $[B \vee C]$

$R(\neg E)$

$[A]$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$R((A \vee E) \supset \neg(\neg B \supset C))$$

$$L(A)$$

$$L((B \wedge D) \vee C)$$

$R(\neg(A \vee E))$ — $R(\neg(\neg B \supset C))$

$R(\neg A)$

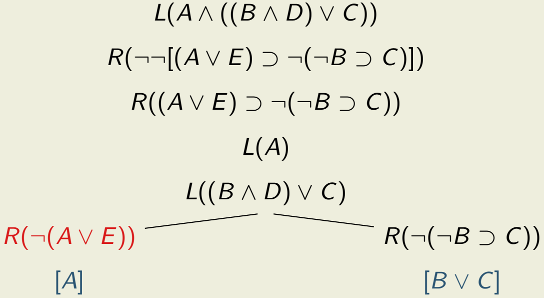
$R(\neg E)$

$[A]$

$[B \vee C]$

Example

interpolant for tautology $[A \land ((B \land D) \lor C] \supset \neg[(A \lor E) \supset \neg(\neg B \supset C)]$



$$L(A \land ((B \land D) \lor C))$$

$$R(\neg\neg[(A \lor E) \supset \neg(\neg B \supset C)])$$

$$R((A \lor E) \supset \neg(\neg B \supset C))$$

$$L(A)$$

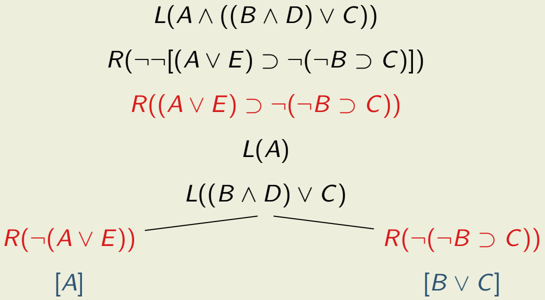
$$L((B \land D) \lor C)$$

$R(\neg(A \lor E))$ $\quad$ $R(\neg(\neg B \supset C))$

$[A]$ $\quad$ $[B \lor C]$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$



$L(A \wedge ((B \wedge D) \vee C))$

$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$

$R((A \vee E) \supset \neg(\neg B \supset C))$

$L(A)$

$L((B \wedge D) \vee C)$

$R(\neg(A \vee E))$ $\qquad$ $R(\neg(\neg B \supset C))$

$[A]$ $\qquad$ $[B \vee C]$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$

$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$R((A \vee E) \supset \neg(\neg B \supset C))$$

$$L(A)$$

$$L((B \wedge D) \vee C)$$

$$[A \wedge (B \vee C)]$$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$

$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$R((A \vee E) \supset \neg(\neg B \supset C))$$

$$L(A)$$

$$L((B \wedge D) \vee C)$$

$$[A \wedge (B \vee C)]$$

Example

interpolant for tautology $[A \land ((B \land D) \lor C] \supset \neg[(A \lor E) \supset \neg(\neg B \supset C)]$

$$L(A \land ((B \land D) \lor C))$$

$$R(\neg\neg[(A \lor E) \supset \neg(\neg B \supset C)])$$

$$R((A \lor E) \supset \neg(\neg B \supset C))$$

$$[A \land (B \lor C)]$$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$

$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$R((A \vee E) \supset \neg(\neg B \supset C))$$

$$[A \wedge (B \vee C)]$$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$

$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$[A \wedge (B \vee C)]$$

Example

interpolant for tautology $[A \wedge ((B \wedge D) \vee C] \supset \neg[(A \vee E) \supset \neg(\neg B \supset C)]$

$$L(A \wedge ((B \wedge D) \vee C))$$

$$R(\neg\neg[(A \vee E) \supset \neg(\neg B \supset C)])$$

$$[A \wedge (B \vee C)]$$

no function symbols

no function symbols

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{L(\delta)\} \xrightarrow{\text{int}} A}$$

provided parameter p does not occur in S or δ

no function symbols

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{L(\delta)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{R(\delta)\} \xrightarrow{\text{int}} A}$$

provided parameter p does not occur in S or δ

no function symbols

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{L(\delta)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{R(\delta)\} \xrightarrow{\text{int}} A}$$

provided parameter p does not occur in S or δ

$$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$$

no function symbols

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{L(\delta)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{R(\delta)\} \xrightarrow{\text{int}} A}$$

provided parameter p does not occur in S or δ

$$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} A}$$

provided constant c occurs in $\{X_1, \ldots, X_n\}$

no function symbols

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{L(\delta)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{R(\delta)\} \xrightarrow{\text{int}} A}$$

provided parameter p does not occur in S or δ

$$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} A}$$

provided constant c occurs in $\{Y_1, \dots, Y_k\}$

no function symbols

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{L(\delta)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\delta(p))\} \xrightarrow{\text{int}} A}{S \cup \{R(\delta)\} \xrightarrow{\text{int}} A}$$

provided parameter p does not occur in S or δ

$$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} A} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} A}$$

provided constant c occurs in $\{X_1, \dots, X_n\}$ / $\{Y_1, \dots, Y_k\}$

$$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \ldots, X_n\}$

$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \ldots, X_n\}$

Notation

$A\{c/x\}$ is result of replacing all occurrences of c in A with x

$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{Y_1, \ldots, Y_k\}$

Notation

$A\{c/x\}$ is result of replacing all occurrences of c in A with x

$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \dots, X_n\}$ / $\{Y_1, \dots, Y_k\}$

Notation

$A\{c/x\}$ is result of replacing all occurrences of c in A with x

$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \dots, X_n\}$ / $\{Y_1, \dots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$

$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \dots, X_n\}$ / $\{Y_1, \dots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma(c))$ are valid

$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \dots, X_n\}$ / $\{Y_1, \dots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c occurs in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma(c))$ are valid
- $\gamma \supset \gamma(c)$ is valid

$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \dots, X_n\}$ / $\{Y_1, \dots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c occurs in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma(c))$ are valid
- $\gamma \supset \gamma(c)$ is valid and hence $\neg\gamma(c) \supset \neg\gamma$ is valid

$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \ldots, X_n\}$ / $\{Y_1, \ldots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c occurs in $\{Y_1, \ldots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma(c))$ are valid
- $\gamma \supset \gamma(c)$ is valid and hence $\neg\gamma(c) \supset \neg\gamma$ is valid
- $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma)$ is valid

$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \ldots, X_n\}$ / $\{Y_1, \ldots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c occurs in $\{Y_1, \ldots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma)$ are valid

$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x) A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x) A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \dots, X_n\}$ / $\{Y_1, \dots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c occurs in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma)$ are valid
- all relation and constant symbols of A occur both in $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_k, \gamma\}$ because c occurs in $\{Y_1, \dots, Y_k\}$

$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \dots, X_n\}$ / $\{Y_1, \dots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c occurs in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma)$ are valid
- all relation and constant symbols of A occur both in $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_k, \gamma\}$ because c occurs in $\{Y_1, \dots, Y_k\}$
- A is interpolant for $S \cup \{R(\gamma)\}$

$$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$$

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \ldots, X_n\}$ / $\{Y_1, \ldots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \ldots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma(c))$ are valid

$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \ldots, X_n\}$ / $\{Y_1, \ldots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \ldots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma(c))$ are valid
- $A \supset (\exists x)A\{c/x\}$ is valid

$S = \{L(X_1), \ldots, L(X_n), R(Y_1), \ldots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \ldots, X_n\}$ / $\{Y_1, \ldots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \ldots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma(c))$ are valid
- $A \supset (\exists x)A\{c/x\}$ is valid and hence $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ is valid

$S = \{L(X_1), \dots, L(X_n), R(Y_1), \dots, R(Y_k)\}$ fresh variable x

Calculation Rules for Interpolants (cont'd)

$$\frac{S \cup \{L(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{L(\gamma)\} \xrightarrow{\text{int}} (\forall x)A\{c/x\}} \qquad \frac{S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A}{S \cup \{R(\gamma)\} \xrightarrow{\text{int}} (\exists x)A\{c/x\}}$$

provided constant c does not occur in $\{X_1, \dots, X_n\}$ / $\{Y_1, \dots, Y_k\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset A$ and $A \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma(c))$ are valid
- $A \supset (\exists x)A\{c/x\}$ is valid and hence $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ is valid
- $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ is valid

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \ldots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ and $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ are valid

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ and $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ are valid
- $(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} \supset (\forall x)\neg A\{c/x\}$ is valid

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \ldots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ and $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ are valid
- $(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} \supset (\forall x)\neg A\{c/x\}$ is valid

$$(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} \equiv Y_1 \wedge \cdots \wedge Y_k \wedge (\forall x)\gamma(c)\{c/x\}$$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ and $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ are valid
- $(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} \supset (\forall x)\neg A\{c/x\}$ is valid

$$\begin{aligned}(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge (\forall x)\gamma(c)\{c/x\} \\ &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge \gamma\end{aligned}$$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ and $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ are valid
- $(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} \supset (\forall x)\neg A\{c/x\}$ is valid

$$\begin{aligned}(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge (\forall x)\gamma(c)\{c/x\} \\ &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge \gamma\end{aligned}$$

- $\neg(\forall x)\neg A\{c/x\} \supset \neg(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma)$ is valid

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ and $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ are valid
- $(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} \supset (\forall x)\neg A\{c/x\}$ is valid

$$\begin{aligned}(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge (\forall x)\gamma(c)\{c/x\} \\ &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge \gamma\end{aligned}$$

- $\neg(\forall x)\neg A\{c/x\} \supset \neg(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma)$ is valid
- $(\exists x)A\{c/x\} \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma)$ is valid

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ and $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ are valid
- $(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} \supset (\forall x)\neg A\{c/x\}$ is valid

$$\begin{aligned}(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge (\forall x)\gamma(c)\{c/x\} \\ &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge \gamma\end{aligned}$$

- $\neg(\forall x)\neg A\{c/x\} \supset \neg(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma)$ is valid
- $(\exists x)A\{c/x\} \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma)$ is valid
- all relation and constant symbols of A occur both in $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_k, \gamma(c)\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ and $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ are valid
- $(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} \supset (\forall x)\neg A\{c/x\}$ is valid

$$\begin{aligned}(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge (\forall x)\gamma(c)\{c/x\} \\ &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge \gamma\end{aligned}$$

- $\neg(\forall x)\neg A\{c/x\} \supset \neg(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma)$ is valid
- $(\exists x)A\{c/x\} \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma)$ is valid
- all relation and constant symbols of A occur both in $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_k, \gamma(c)\}$
- all relation and constant symbols of $(\exists x)A\{c/x\}$ occur both in $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_k, \gamma\}$

Verification (cont'd)

suppose $S \cup \{R(\gamma(c))\} \xrightarrow{\text{int}} A$ and c does not occur in $\{Y_1, \dots, Y_k\}$

- $(X_1 \wedge \cdots \wedge X_n) \supset (\exists x)A\{c/x\}$ and $(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)) \supset \neg A$ are valid
- $(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} \supset (\forall x)\neg A\{c/x\}$ is valid

$$\begin{aligned}(\forall x)[Y_1 \wedge \cdots \wedge Y_k \wedge \gamma(c)]\{c/x\} &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge (\forall x)\gamma(c)\{c/x\} \\ &\equiv Y_1 \wedge \cdots \wedge Y_k \wedge \gamma\end{aligned}$$

- $\neg(\forall x)\neg A\{c/x\} \supset \neg(Y_1 \wedge \cdots \wedge Y_k \wedge \gamma)$ is valid
- $(\exists x)A\{c/x\} \supset (\neg Y_1 \vee \cdots \vee \neg Y_k \vee \neg\gamma)$ is valid
- all relation and constant symbols of A occur both in $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_k, \gamma(c)\}$
- all relation and constant symbols of $(\exists x)A\{c/x\}$ occur both in $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_k, \gamma\}$
- $(\exists x)A\{c/x\}$ is interpolant for $S \cup \{R(\gamma)\}$

Outline

- Overview of this lecture
- Transforming Hilbert Style proof into tableau proof with cut
- Gentzen's Hauptsatz: Cut Elimination
- Craig's Interpolation Theorem
- Prenex Form
- Exercises
- Further Reading

Definition

formula Φ has its variables named apart if no two quantifiers in Φ bind same variable and no bound variable is also free

Definition

formula Φ has its variables named apart if no two quantifiers in Φ bind same variable and no bound variable is also free

Quantifier Rewrite Rules

$$\neg(\exists x)A \equiv (\forall x)\neg A \qquad \neg(\forall x)A \equiv (\exists x)\neg A$$

Definition

formula Φ has its variables named apart if no two quantifiers in Φ bind same variable and no bound variable is also free

Quantifier Rewrite Rules

$$\begin{aligned} \neg(\exists x)A &\equiv (\forall x)\neg A \\ [(\forall x)A \wedge B] &\equiv (\forall x)[A \wedge B] \\ [A \wedge (\forall x)B] &\equiv (\forall x)[A \wedge B] \end{aligned}$$

$$\neg(\forall x)A \equiv (\exists x)\neg A$$

Definition

formula Φ has its variables named apart if no two quantifiers in Φ bind same variable and no bound variable is also free

Quantifier Rewrite Rules

$$\begin{aligned} \neg(\exists x)A &\equiv (\forall x)\neg A \\ [(\forall x)A \wedge B] &\equiv (\forall x)[A \wedge B] \\ [A \wedge (\forall x)B] &\equiv (\forall x)[A \wedge B] \\ [(\exists x)A \wedge B] &\equiv (\exists x)[A \wedge B] \\ [A \wedge (\exists x)B] &\equiv (\exists x)[A \wedge B] \end{aligned}$$

$$\neg(\forall x)A \equiv (\exists x)\neg A$$

Definition

formula Φ has its variables named apart if no two quantifiers in Φ bind same variable and no bound variable is also free

Quantifier Rewrite Rules

$$\begin{aligned} \neg(\exists x)A &\equiv (\forall x)\neg A \\ [(\forall x)A \wedge B] &\equiv (\forall x)[A \wedge B] \\ [A \wedge (\forall x)B] &\equiv (\forall x)[A \wedge B] \\ [(\exists x)A \wedge B] &\equiv (\exists x)[A \wedge B] \\ [A \wedge (\exists x)B] &\equiv (\exists x)[A \wedge B] \end{aligned}$$

$$\begin{aligned} \neg(\forall x)A &\equiv (\exists x)\neg A \\ [(\forall x)A \supset B] &\equiv (\exists x)[A \supset B] \\ [A \supset (\forall x)B] &\equiv (\forall x)[A \supset B] \end{aligned}$$

Definition

formula Φ has its variables named apart if no two quantifiers in Φ bind same variable and no bound variable is also free

Quantifier Rewrite Rules

$$\begin{aligned}
\neg(\exists x)A &\equiv (\forall x)\neg A \\
[(\forall x)A \wedge B] &\equiv (\forall x)[A \wedge B] \\
[A \wedge (\forall x)B] &\equiv (\forall x)[A \wedge B] \\
[(\exists x)A \wedge B] &\equiv (\exists x)[A \wedge B] \\
[A \wedge (\exists x)B] &\equiv (\exists x)[A \wedge B]
\end{aligned}
\qquad
\begin{aligned}
\neg(\forall x)A &\equiv (\exists x)\neg A \\
[(\forall x)A \supset B] &\equiv (\exists x)[A \supset B] \\
[A \supset (\forall x)B] &\equiv (\forall x)[A \supset B] \\
[(\exists x)A \supset B] &\equiv (\forall x)[A \supset B] \\
[A \supset (\exists x)B] &\equiv (\exists x)[A \supset B]
\end{aligned}$$

Definition

formula Φ has its variables named apart if no two quantifiers in Φ bind same variable and no bound variable is also free

Quantifier Rewrite Rules

$$\begin{aligned} \neg(\exists x)A &\equiv (\forall x)\neg A \\ [(\forall x)A \wedge B] &\equiv (\forall x)[A \wedge B] \\ [A \wedge (\forall x)B] &\equiv (\forall x)[A \wedge B] \\ [(\exists x)A \wedge B] &\equiv (\exists x)[A \wedge B] \\ [A \wedge (\exists x)B] &\equiv (\exists x)[A \wedge B] \\ &\cdots \end{aligned} \qquad \begin{aligned} \neg(\forall x)A &\equiv (\exists x)\neg A \\ [(\forall x)A \supset B] &\equiv (\exists x)[A \supset B] \\ [A \supset (\forall x)B] &\equiv (\forall x)[A \supset B] \\ [(\exists x)A \supset B] &\equiv (\forall x)[A \supset B] \\ [A \supset (\exists x)B] &\equiv (\exists x)[A \supset B] \\ &\cdots \end{aligned}$$

Example

$$(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y)$$

Example

$$\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y) \\
&\quad \equiv\ (\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)
\end{aligned}$$

Example

$$
\begin{aligned}
(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y) & \\
\equiv\ & (\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w) \\
\equiv\ & (\forall x)[(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)]
\end{aligned}
$$

Example

$$
\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y)\\
&\qquad\equiv\ (\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)\\
&\qquad\equiv\ (\forall x)[(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)]\\
&\qquad\equiv\ (\forall x)(\exists y)[R(x,y) \supset (\forall z)(\exists w)R(z,w)]
\end{aligned}
$$

Example

$$\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y)\\
&\qquad \equiv\ (\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)\\
&\qquad \equiv\ (\forall x)[(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)]\\
&\qquad \equiv\ (\forall x)(\exists y)[R(x,y) \supset (\forall z)(\exists w)R(z,w)]\\
&\qquad \equiv\ (\forall x)(\exists y)(\forall z)[R(x,y) \supset (\exists w)R(z,w)]
\end{aligned}$$

Example

$$\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y)\\
&\qquad \equiv \ (\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)\\
&\qquad \equiv \ (\forall x)[(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)]\\
&\qquad \equiv \ (\forall x)(\exists y)[R(x,y) \supset (\forall z)(\exists w)R(z,w)]\\
&\qquad \equiv \ (\forall x)(\exists y)(\forall z)[R(x,y) \supset (\exists w)R(z,w)]\\
&\qquad \equiv \ (\forall x)(\exists y)(\forall z)(\exists w)[R(x,y) \supset R(z,w)]
\end{aligned}$$

Example

$$
\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y)\\
&\qquad \equiv \; (\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)\\
&\qquad \equiv \; (\forall x)[(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)]\\
&\qquad \equiv \; (\forall x)(\exists y)[R(x,y) \supset (\forall z)(\exists w)R(z,w)]\\
&\qquad \equiv \; (\forall x)(\exists y)(\forall z)[R(x,y) \supset (\exists w)R(z,w)]\\
&\qquad \equiv \; (\forall x)(\exists y)(\forall z)(\exists w)[R(x,y) \supset R(z,w)]
\end{aligned}
$$

$$
\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset {\color{red}(\forall z)}(\exists w)R(z,w)\\
&\qquad \equiv \; (\forall z)[(\exists x)(\forall y)R(x,y) \supset (\exists w)R(z,w)]
\end{aligned}
$$

Example

$$
\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y) \\
&\qquad \equiv \ (\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w) \\
&\qquad \equiv \ (\forall x)[(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)] \\
&\qquad \equiv \ (\forall x)(\exists y)[R(x,y) \supset (\forall z)(\exists w)R(z,w)] \\
&\qquad \equiv \ (\forall x)(\exists y)(\forall z)[R(x,y) \supset (\exists w)R(z,w)] \\
&\qquad \equiv \ (\forall x)(\exists y)(\forall z)(\exists w)[R(x,y) \supset R(z,w)]
\end{aligned}
$$

$$
\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w) \\
&\qquad \equiv \ (\forall z)[(\exists x)(\forall y)R(x,y) \supset (\exists w)R(z,w)] \\
&\qquad \equiv \ (\forall z)(\exists w)[(\exists x)(\forall y)R(x,y) \supset R(z,w)]
\end{aligned}
$$

Example

$$
\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y)\\
&\qquad \equiv \; (\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)\\
&\qquad \equiv \; (\forall x)[(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)]\\
&\qquad \equiv \; (\forall x)(\exists y)[R(x,y) \supset (\forall z)(\exists w)R(z,w)]\\
&\qquad \equiv \; (\forall x)(\exists y)(\forall z)[R(x,y) \supset (\exists w)R(z,w)]\\
&\qquad \equiv \; (\forall x)(\exists y)(\forall z)(\exists w)[R(x,y) \supset R(z,w)]
\end{aligned}
$$

$$
\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)\\
&\qquad \equiv \; (\forall z)[(\exists x)(\forall y)R(x,y) \supset (\exists w)R(z,w)]\\
&\qquad \equiv \; (\forall z)(\exists w)[{\color{red}(\exists x)}(\forall y)R(x,y) \supset R(z,w)]\\
&\qquad \equiv \; (\forall z)(\exists w)(\forall x)[(\forall y)R(x,y) \supset R(z,w)]
\end{aligned}
$$

Example

$$\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall y)(\exists x)R(x,y) \\
&\qquad \equiv \ (\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w) \\
&\qquad \equiv \ (\forall x)[(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w)] \\
&\qquad \equiv \ (\forall x)(\exists y)[R(x,y) \supset (\forall z)(\exists w)R(z,w)] \\
&\qquad \equiv \ (\forall x)(\exists y)(\forall z)[R(x,y) \supset (\exists w)R(z,w)] \\
&\qquad \equiv \ (\forall x)(\exists y)(\forall z)(\exists w)[R(x,y) \supset R(z,w)]
\end{aligned}$$

$$\begin{aligned}
&(\exists x)(\forall y)R(x,y) \supset (\forall z)(\exists w)R(z,w) \\
&\qquad \equiv \ (\forall z)[(\exists x)(\forall y)R(x,y) \supset (\exists w)R(z,w)] \\
&\qquad \equiv \ (\forall z)(\exists w)[(\exists x)(\forall y)R(x,y) \supset R(z,w)] \\
&\qquad \equiv \ (\forall z)(\exists w)(\forall x)[{\color{red}(\forall y)}R(x,y) \supset R(z,w)] \\
&\qquad \equiv \ (\forall z)(\exists w)(\forall x)(\exists y)[R(x,y) \supset R(z,w)]
\end{aligned}$$

Definition

prenex form is formula $(Q_1x_1)\dots(Q_nx_n)\Phi$ with $Q_i \in \{\forall, \exists\}$ for all $1 \leqslant i \leqslant n$ and Φ quantifier-free, its matrix

Definition

prenex form is formula $(Q_1 x_1) \dots (Q_n x_n)\Phi$ with $Q_i \in \{\forall, \exists\}$ for all $1 \leqslant i \leqslant n$ and Φ quantifier-free, its matrix

Lemma

for every quantified formula X there exists equivalent prenex form X'

Definition

prenex form is formula $(Q_1 x_1) \ldots (Q_n x_n)\Phi$ with $Q_i \in \{\forall, \exists\}$ for all $1 \leqslant i \leqslant n$ and Φ quantifier-free, its matrix

Lemma

for every quantified formula X there exists equivalent prenex form X'

Proof

1. rename all bound variables such that every quantifier binds unique variable

Definition

prenex form is formula $(Q_1 x_1) \dots (Q_n x_n)\Phi$ with $Q_i \in \{\forall, \exists\}$ for all $1 \leqslant i \leqslant n$ and Φ quantifier-free, its matrix

Lemma

for every quantified formula X there exists equivalent prenex form X'

Proof

1. rename all bound variables such that every quantifier binds unique variable
2. Quantifier Rewrite Rules push propositional connectives through quantifiers

Definition

prenex form is formula $(Q_1 x_1) \dots (Q_n x_n)\Phi$ with $Q_i \in \{\forall, \exists\}$ for all $1 \leqslant i \leqslant n$ and Φ quantifier-free, its matrix

Lemma

for every quantified formula X there exists equivalent prenex form X'

Proof

1. rename all bound variables such that every quantifier binds unique variable
2. Quantifier Rewrite Rules push propositional connectives through quantifiers

Corollary

there exists algorithm for converting sentence Φ into sentence Φ^* in prenex form with only universal quantifiers such that $\{\Phi\}$ is satisfiable if and only if $\{\Phi^*\}$ is satisfiable

Outline

- Overview of this lecture
- Transforming Hilbert Style proof into tableau proof with cut
- Gentzen's Hauptsatz: Cut Elimination
- Craig's Interpolation Theorem
- Prenex Form
- Exercises
- Further Reading

Fitting

- Exercise 8.4.2
- Exercise 8.9.1
- Exercise 8.11.1
- Complete the example on page 260, indicate the steps taken.
- Exercise 8.12.1
- Exercise 8.12.2
- Bonus. Solving some of above exercises by means of an implementation:
    - Exercise 8.4.2 with solution: 1 additional cross
    - Exercise 8.9.1 with solution: 3 additional crosses for propositional case; 4 more for first-order case
    - Exercise 8.12.1: 3 additional crosses for propositional case (starting from some tableau proof); 4 more for first-order case with solution

At most one of the last two bonus items may be chosen.

Outline

- Overview of this lecture
- Transforming Hilbert Style proof into tableau proof with cut
- Gentzen's Hauptsatz: Cut Elimination
- Craig's Interpolation Theorem
- Prenex Form
- Exercises
- Further Reading

Fitting

- Section 8.4
- Section 8.8
- Section 8.9
- Section 8.10 (only page 243, as background information)
- Section 8.11
- Section 8.12

Additional material

For more background and motivation on first order model theory (compactness, Löwenheim–Skolem, interpolation) or proof theory (Gentzen's cut-elimination), see e.g. the Stanford Encyclopedia of Philosophy.