universität innsbruck

Program Verification	SS 2021	LVA 703083+703084
Sheet 5		Deadline: April 20, 2021, 8am

- Prepare your solutions on paper.
- Marking an exercise in OLAT means that a significant part of that exercise has been treated.
- Upload your Haskell files and your paper solution in OLAT, the latter as one PDF.

Exercise 1 *Matching Algorithm* 7 p.

The matching algorithm has been proven correct in the lecture. However, the algorithm itself is only pseudo-code.

1. Implement the matching algorithm in Haskell. A template-file is given. (3 points)
2. Implement an algorithm in Haskell which evaluates a term t one step, i.e., either some term s such that $t \hookrightarrow s$ should be returned, or it should be indicated that there is no such term. A template-file is given. (3 points)
3. Specify soundness and completeness properties of your algorithm, without proving them. (1 point)

Exercise 2 *Confluence and Normal Forms* 4 p.

1. Recall the notion of confluence: $\hookrightarrow$ is confluent if and only if for all terms s, t, u: whenever $t \hookrightarrow^* s$ and $t \hookrightarrow^* u$ then there is some term v such that $s \hookrightarrow^* v$ and $u \hookrightarrow^* v$.

   Prove that normal forms are unique for confluent $\hookrightarrow$: whenever $\hookrightarrow$ is confluent and $t \hookrightarrow^! s$ and $t \hookrightarrow^! u$, then $s = u$. (2 points)
2. On slide 3/56 it was stated that $\ell\alpha\downarrow = r\alpha\downarrow$ follows from $\ell\alpha \hookrightarrow r\alpha$.

   Perform a proof of a more general result: whenever $s \hookrightarrow^* t$ for some confluent and terminating relation $\hookrightarrow$, then $s\downarrow = t\downarrow$. (2 points)

Exercise 3 *Non-Pattern-Disjoint Programs* 3 p.

On slides 3/47-48 it was shown that every functional program that is not pattern disjoint can be transformed into an equivalent pattern disjoint program by taking the order of the equations into account.
In this exercise you should investigate what kind of consequences this has for a potential user. To this end consider the following program $\mathcal{P}$:

$$\begin{aligned} \mathsf{le}(\mathsf{Succ}(x), \mathsf{Succ}(y)) &= \mathsf{le}(x, y) \\ \mathsf{le}(\mathsf{Succ}(x), \mathsf{Zero}) &= \mathsf{False} \\ \mathsf{le}(x, y) &= \mathsf{True} \end{aligned}$$

1. Translate $\mathcal{P}$ into an equivalent pattern disjoint program $\mathcal{P}'$ by taking the order of equations into account. (1 point)
2. Compare the theorems that you get from the defining equations of $\mathcal{P}$ and of $\mathcal{P}'$ w.r.t. slide 3/56. Can the transformation be revealed from the user in the same was as it was done in the case of pattern completeness? (2 points)

Exercise 4 *Axioms about Equality* **6 p.**

On slide 3/60 we have seen the disjointness and decomposition theorems about $=_\tau$.
In this exercise we will derive further axioms about $=_\tau$ namely that $=_\tau$ is an equivalence relation (we only show symmetry in this exercise) and a congruence.

1. Prove that $=_\tau$ is symmetric, i.e., $\mathcal{M} \models \forall x.\, x =_\tau y \longleftrightarrow y =_\tau x$ for the standard model $\mathcal{M}$. (3 points)
2. Prove that $=_\tau$ is a congruence, i.e., whenever $f : \tau_1 \times \ldots \times \tau_n \to \tau \in \Sigma$ then

$$\mathcal{M} \models \forall x_1, y_1, \ldots, x_n, y_n.\, x_1 =_{\tau_1} y_1 \longrightarrow \ldots \longrightarrow x_n =_{\tau_n} y_n \longrightarrow f(x_1, \ldots, x_n) =_\tau f(y_1, \ldots, y_n)$$

   for the standard model $\mathcal{M}$. (3 points)