- Prepare your solutions on paper.
- Marking an exercise in OLAT means that a significant part of that exercise has been treated.
- Upload your solution in OLAT as a single PDF file.

Exercise 1 *Ramsey's Theorem* **3 p.**

Let $G = (V, E)$ be an infinite undirected graph without self-loops, i.e., $|V| = \infty$ and $E \subseteq V \times V$ is the edge relation, where $(v, v) \notin E$ for all $v \in V$. Show that G contains an infinite subgraph G' that is either fully connected or not connected at all, i.e., there exists $W \subseteq V$, $|W| = \infty$ and $G' = (W, E \cap W \times W)$ is either fully connected or empty.
You can prove the result with the help of Ramsey's theorem. In your argumentation clarify how you choose the parameters X, c, n, C within Ramsey's theorem. (3 points)

Exercise 2 *Computation of Multigraphs* **6 p.**

In the lecture the set of multigraphs $\mathcal{M}$ of a set of size-change graphs $\mathcal{G}$ has essentially been defined as follows:

$$\frac{G \in \mathcal{G}}{G \in \mathcal{M}} \qquad\qquad \frac{G_1 \in \mathcal{M} \quad G_2 \in \mathcal{M}}{G_1 \bullet G_2 \in \mathcal{M}}$$

Now consider the following set of multigraphs $\mathcal{N}$, defined as:

$$\frac{G \in \mathcal{G}}{G \in \mathcal{N}} \qquad\qquad \frac{G_1 \in \mathcal{G} \quad G_2 \in \mathcal{N}}{G_1 \bullet G_2 \in \mathcal{N}}$$

In this exercise we will show that both definitions are equivalent. The advantage of using $\mathcal{N}$ is the following: it is faster to compute $\mathcal{N}$, since one always just has to combine each newly found multigraph with all original graphs in $\mathcal{G}$, but not with all other newly detected graphs as in the definition of $\mathcal{M}$.

1. Prove $\mathcal{N} \subseteq \mathcal{M}$. (2 points)
2. Prove $\mathcal{M} \subseteq \mathcal{N}$. You can assume that $\bullet$ is associative. Most likely, you will need to prove one auxiliary property. (4 points)

Exercise 3 *Size-Change Termination* **6 p.**

Consider the following set of dependency pairs:

$$\mathsf{f}^\sharp(m, n, \mathsf{S}(r)) \to \mathsf{f}^\sharp(m, r, n)$$
$$\mathsf{f}^\sharp(m, \mathsf{S}(n), r) \to \mathsf{f}^\sharp(r, n, m)$$

Turn these dependency pairs into size-change graphs, and figure out whether termination can be proved by via size-change termination.

Important: you should of course use the optimization from Exercise 2, but you may also use the following optimization. Whenever you encounter a multigraph which has strictly more information than another one, then the multigraph with more information can be completely ignored (in particular while trying to computate new multigraphs). Here, G has more information than G' (written $G \supseteq G'$) whenever all edges of G' are also present in G, and whenever an edge in G' has a label with strict decrease, then the same edge in G is also labelled with a strict decrease. Example: $G_1 = \{1 \overset{\succ}{\to} 1, 1 \overset{\succ}{\to} 2, 2 \overset{\succsim}{\to} 3\} \supseteq \{1 \overset{\succsim}{\to} 1, 2 \overset{\succsim}{\to} 3\} = G_2$. Hence, if G_2 is already a multigraph and G_1 is newly created, then G_1 can be ignored for computing new multigraphs.
Remark: If during the process of generating multigraphs you have performed more than 20 concatenations of multigraphs, you may stop and still mark this exercise in OLAT. (The calculation in the solution requires 16 concatenations and results in 8 multigraphs in the closure. But the calculation might be wrong, since it was done manually.)

(4 points)

Exercise 4 *Polynomial Interpretations* **5 p.**

Consider the following functional program for computing the binary logarithm.

$$\begin{aligned}
\mathsf{half}(\mathsf{Zero}) &= \mathsf{Zero}\\
\mathsf{half}(\mathsf{Succ}(\mathsf{Zero})) &= \mathsf{Zero}\\
\mathsf{half}(\mathsf{Succ}(\mathsf{Succ}(x))) &= \mathsf{Succ}(\mathsf{half}(x))\\
\mathsf{log2}(\mathsf{Zero}) &= \mathsf{Zero}\\
\mathsf{log2}(\mathsf{Succ}(\mathsf{Zero})) &= \mathsf{Zero}\\
\mathsf{log2}(\mathsf{Succ}(\mathsf{Succ}(x))) &= \mathsf{Succ}(\mathsf{log2}(\mathsf{Succ}(\mathsf{half}(x))))
\end{aligned}$$

1. Write down all dependency pairs that cannot be solved by the subterm criterion and determine the usable equations for these dependency pairs. (1 point)
2. Prove termination via polynomial interpretations. First setup the constraints symbolically, and then choose between manual solving and SMT-solving. For the latter you can either directly download and compile Z3 from github, or use a binary version that is distributed as part of Isabelle in the `contrib/z3...` directory. (4 points)