universität innsbruck

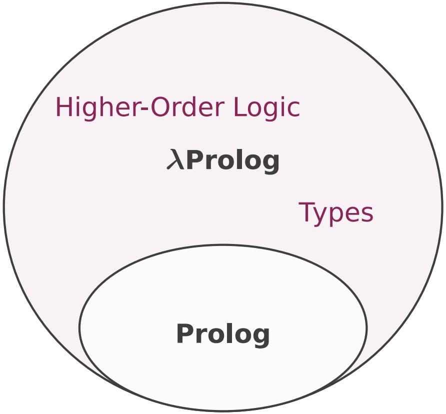
λProlog: Higher-Order Logic Programming

Specialisation Seminar

Jamie Fox

Introduction

- Prolog is a **logic programming** language
- λProlog extends Prolog



Introduction

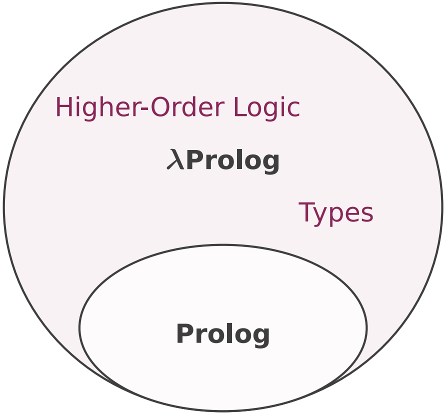
- Prolog is a **logic programming** language
- λProlog extends Prolog
- This talk covers...

  - Logic Programming

  - Prolog

  - λProlog

  - Applications



Predicate Logic In 60 Seconds

Terms

Entity	Examples
Variables	`X, Y, Person`
Constants	`p, cat, anne`
Predicates	`mammal(cat), parent(anne,Person), f(X,g(Y))`

Predicate Logic In 60 Seconds

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Logic Operations

$p \wedge q$	p and q
$p \vee q$	p or q
$p \leftarrow q$	p if q (“implied by”)
$\exists X\ p$	There exists an X such that p
$\forall X\ p$	For all X, p

Logic Programming

- Programming paradigm based on formal logic
- Programs consist of facts and rules

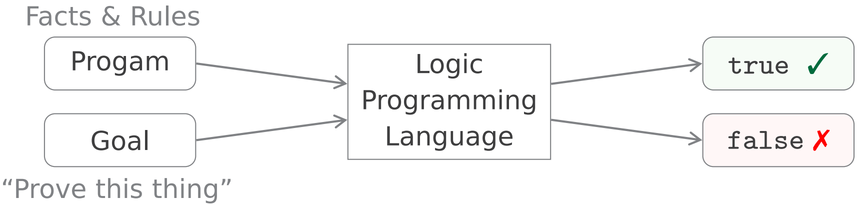
Example

Facts: $\texttt{p(X)}, \texttt{parent(anne, bob)}, \texttt{even(2)}$

Rules: $\texttt{sibling(X, Y)} \Longleftarrow \texttt{parent(P, X)} \land \texttt{parent(P, Y)}$

Logic Programming

- Programming paradigm based on formal logic
- Programs consist of facts and rules
- Typically have a logic program and try to prove a **goal**



Motivation

Why program using formal logic?

1. **Proof Formalisation**
   - Humans can make mistakes in proofs, computers (hopefully!) reason correctly
2. **Program Verification**
   - Testing can not *prove* that code works as intended – we need logic
3. **Modelling Algorithms**
   - Natural language processing, compiler design, constraint solving, ...

1. Introduction
2. Logic Programming
3. **Prolog**

   Background

   Horn Clauses

   Examples
4. λProlog
5. Applications

Prolog – Background

- Developed in France in $\sim$1972
- Prolog is a **declarative** language
    - Tell Prolog what to prove but not *how* to prove it

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Horn Clauses

Define goal formulas $\mathtt{G}$ and program clauses $\mathtt{D}$ as follows:

$$\mathtt{G} ::= \top \mid \mathtt{A} \mid \mathtt{G} \land \mathtt{G} \mid \mathtt{G} \lor \mathtt{G} \mid \exists \mathtt{X}\ \mathtt{G}$$

$$\mathtt{D} ::= \mathtt{A} \mid \mathtt{G} \rightarrow \mathtt{D} \mid \mathtt{D} \land \mathtt{D} \mid \forall \mathtt{X}\ \mathtt{D}$$

where A is a first-order atomic formula and quantification is over variables.

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Formula	$\mathtt{D}$	$\mathtt{G}$
$p \vee (q \wedge r)$	✗	✓
$\exists X\ f(X) \wedge g(X)$	✗	✓

} No ∨ or ∃ at the top level in programs

Horn Clause Examples

Horn Clauses

Define goal formulas G and program clauses D as follows:

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Formula	D	G	
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$\exists X\ f(X) \wedge g(X)$	✗	✓	
$\forall X\ f(X)$	✓	✗	No ∀ or → goals
$p \wedge q \rightarrow p$	✓	✗	

Horn Clause Examples

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$\forall X\ f(X)$	✓	✗	No ∀ or → goals
$p \wedge q \to p$	✓	✗	
$\forall F\ g(F(x))$	✗	✗	No quantification over functions

Prolog – Read, Prove, Print Loop

- Each clause ends with a “.”
- Prolog demos also work in λProlog

Program

```
owns(alex, cat(fluffy)).      % alex owns a cat called fluffy
owns(alex, dog(frodo)).
owns(dave, cat(whiskers)).
```

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```
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Query

```
?- owns(alex, Pet).           % Does alex own any pets?
```

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Pet = cat(fluffy) .
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Pet = cat(fluffy) .

?- owns(dave, cat(X)).          % Does dave own any cats?
```

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X = whiskers.
```

Prolog Demo

In Prolog clauses, “`:-`” means “$\leftarrow$” and “`,`” means “$\wedge$”

Program

```
% anne has children bob and cara
parent(anne, bob).
parent(anne, cara).
sibling(X, Y) :- parent(P, X), parent(P, Y), not(X = Y).
```

Prolog Demo

In Prolog clauses, “`:-`” means “←” and “`,`” means “$\wedge$”

Program

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% anne has children bob and cara
parent(anne, bob).
parent(anne, cara).
sibling(X, Y) :- parent(P, X), parent(P, Y), not(X = Y).
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Query

```
?- sibling(X,Y).
```

How many solutions are there?

Prolog – Multiple Solutions

Program

```
parent(anne, bob).
parent(anne, cara).
sibling(X, Y) :- parent(P, X), parent(P, Y), not(X = Y).
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Query

```
?- sibling(X,Y).
X = bob,
Y = cara
```

Prolog – Multiple Solutions

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parent(anne, bob).
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sibling(X, Y) :- parent(P, X), parent(P, Y), not(X = Y).
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Query

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?- sibling(X,Y).
X = bob,
Y = cara ;      % Prolog can output multiple solutions using ;
X = cara,
Y = bob
```

Prolog – Multiple Solutions

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```
parent(anne, bob).
parent(anne, cara).
sibling(X, Y) :- parent(P, X), parent(P, Y), not(X = Y).
```

Query

```
?- sibling(X,Y).
X = bob,
Y = cara ;       % Prolog can output multiple solutions using ;
X = cara,
Y = bob ;
false.
```

Negation As Failure

Program

```
parent(anne, bob).
parent(dave, cara).
sibling(X, Y) :- parent(P, X), parent(P, Y), not(X = Y).
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Query

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?- sibling(X,Y).
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sibling(X, Y) :- parent(P, X), parent(P, Y), not(X = Y).
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Query

```
?- sibling(X,Y).
false.
```

Provability for first-order Horn clauses is undecidable.
So how did Prolog decide that there are no solutions?

Negation As Failure

Program

```
parent(anne, bob).
parent(dave, cara).
sibling(X, Y) :- parent(P, X), parent(P, Y), not(X = Y).
```

Query

```
?- sibling(X,Y).
false.
```

If Prolog fails to find a solution to a query then it is **assumed to be false**

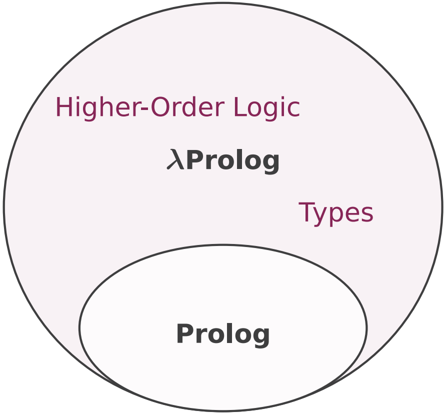
- e.g. we did not specify whether `parent(anne, cara)` is true or false, so Prolog assumes this to be false

What’s Missing?

1. **Higher-Order Logic**

   First order Horn clauses do not allow...

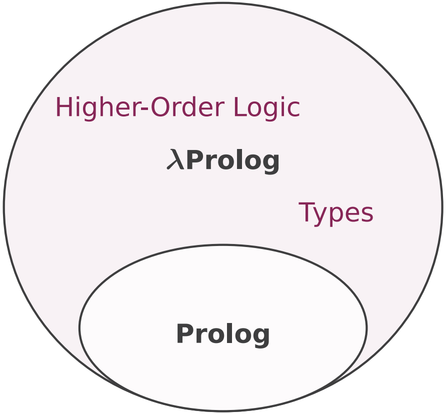
   - Quantification over functions
   - $\rightarrow$ and $\forall$ in goal clauses



What’s Missing?

1. **Higher-Order Logic**
   First order Horn clauses do not allow...
   - Quantification over functions
   - $\rightarrow$ and $\forall$ in goal clauses

2. **Typed Logic Programming**
   What if we want to restrict predicates to certain types of objects?



1. Introduction

2. Logic Programming

3. Prolog

4. λProlog

- Syntax
- Semantics
- Types
- Demo

5. Applications

λProlog Background

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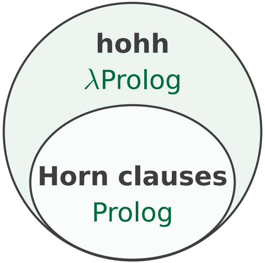
- Adds higher-order logic, native lambda expressions, and polymorphic typing to Prolog
- Based on **higher-order hereditary Harrop** formulas (hohh)



Horn Clauses vs. hohh

hohh

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where elements of $\mathtt{A}$ are normal forms wrt λ-calculus.

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} Still no $\vee$ or $\exists$ at the top level in programs

} $\forall$ and $\rightarrow$ are now allowed in goals

} Quantification is now allowed over functions

1. Introduction

2. Logic Programming

3. Prolog

4. λProlog

- Syntax
- Semantics
- Types
- Demo

5. Applications

Semantics

How does λProlog find solutions?

- λProlog uses a deterministic proof search
  1. Start from the goal query
  2. Work backwards using the current program state (facts and rules)
  3. If stuck:
     - a) Try to backtrack to an earlier proof state
     - b) If no more backtracking is possible, then return `false`

Semantics

How does λProlog find solutions?

- λProlog uses a deterministic proof search
- Variables are instantiated using unification
  - Decidable for first-order logic but undecidable for higher-order logic (beyond the scope of this talk)

Semantics

How does λProlog find solutions?

- λProlog uses a deterministic proof search
- Variables are instantiated using unification
- λProlog is based on **intuitionistic logic**
  - The logic taught in the 4th semester is **classical logic**
  - Classical logic assumes the Law of the Excluded Middle, i.e. that $p \vee \neg p$ always holds

Semantics

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- λProlog is based on **intuitionistic logic**
  - The logic taught in the 4th semester is **classical logic**
  - Classical logic assumes the Law of the Excluded Middle, i.e. that $p \vee \neg p$ always holds
  - Inference in λProlog is unsound for classical logic
  - Solution: get rid of the LEM!

Proof Search

- λProlog uses a deterministic, **depth-first** proof search

Program

```
loop :- loop.
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Proof Search

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Query

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?- loop, fail.      % query 2
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Proof Search

- λProlog uses a deterministic, **depth-first** proof search
- Query 1 and query 2 are logically equivalent
- Semantically, query 1 fails immediately while query 2 will not terminate
- In practice: limit proof search depth

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```
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Query

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?- loop, fail.       % query 2
```

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Syntax

Semantics

Types

Demo

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λProlog Types

λProlog is strongly typed

- Based on simply-typed λ calculus
- Built-in sorts (`int`, `real`, `string`, `list` ...)
- Formulas have special sort `o`

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Example

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Example

- The predicate `sibling X Y` might have type `person -> person -> o`
- A predicate `member X L` which encodes that `X` is a member of list `L` might have type `A -> (list A) -> o` where `A` is a type variable

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λProlog is strongly typed

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Example

- The predicate `sibling X Y` might have type `person -> person -> o`
- A predicate `member X L` which encodes that `X` is a member of list `L` might have type `A -> (list A) -> o` where `A` is a type variable
- “`,`” (logical conjunction) has type `o -> o -> o`

λProlog Demo – 1

- Define new types with `kind`

Program

```
kind dragon_ty                  type.
kind cave_ty                    type.
```

λProlog Demo – 1

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kind dragon_ty, cave_ty         type.
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λProlog Demo – 1

- Define new types with `kind` and value constructors with `type`

Program

```
kind dragon_ty, cave_ty         type.
type asleep, dangerous          dragon_ty -> o.
```

λProlog Demo – 1

- Define new types with `kind` and value constructors with `type`

Program

```
kind dragon_ty, cave_ty         type.
type asleep, dangerous          dragon_ty -> o.
type safe, daytime              cave_ty -> o.
```

λProlog Demo – 1

- Define new types with `kind` and value constructors with `type`

Program

```
kind dragon_ty, cave_ty         type.
type asleep, dangerous          dragon_ty -> o.
type safe, daytime              cave_ty -> o.
type in                         dragon_ty -> cave_ty -> o.
```

λProlog Demo – 1

- Define new types with `kind` and value constructors with `type`
- Dragons are nocturnal
    - If it is daytime in a cave `C` which contains a dragon `D`, then the dragon is asleep

Program

```
kind dragon_ty, cave_ty         type.
type asleep, dangerous          dragon_ty -> o.
type safe, daytime              cave_ty -> o.
type in                         dragon_ty -> cave_ty -> o.

asleep D :- daytime C, in D C.
```

λProlog Demo – 1

- Define new types with `kind` and value constructors with `type`
- Dragons are nocturnal
  - If it is daytime in a cave `C` which contains a dragon `D`, then the dragon is asleep
- A cave is safe if all dangerous dragons in the cave are asleep

Program

```
kind dragon_ty, cave_ty         type.
type asleep, dangerous          dragon_ty -> o.
type safe, daytime              cave_ty -> o.
type in                         dragon_ty -> cave_ty -> o.

asleep D :- daytime C, in D C.
safe C  :- pi D\ (dangerous D,  in D C) => asleep D.
```

λProlog Demo – 2

Program

```
kind dragon_ty, cave_ty         type.
type asleep, dangerous          dragon_ty -> o.
type safe, daytime              cave_ty -> o.
type in                         dragon_ty -> cave_ty -> o.

asleep D :- daytime C, in D C.
safe C  :- pi D\ (dangerous D,  in D C) => asleep D.
```

Query

Logically, it follows that $\forall$`C daytime C` $\to$ `safe C`. We can ask λProlog:

```
?- pi C\ daytime C => safe C.
```

1. **Introduction**
2. **Logic Programming**
3. **Prolog**
4. **λProlog**
5. **Applications**
    - Comparison: λProlog vs. Haskell
    - Who Uses λProlog?
    - Alternatives to λProlog

λProlog vs. Haskell

Similarities

- Both λProlog and Haskell are based on λ calculus and higher-order logic
- Both support polymorphic types, abstraction via modules, and I/O streams

λProlog vs. Haskell

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Differences

1. **Purpose**
   - λProlog has native support for logical inference
   - Haskell is better for general-purpose calculation

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2. **Evaluation**
   - λProlog works backwards from a goal using unification
   - Haskell works forwards using matching-driven lazy evaluation

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Differences

1. **Purpose**
   - λProlog has native support for logical inference
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2. **Evaluation**
   - λProlog works backwards from a goal using unification
   - Haskell works forwards using matching-driven lazy evaluation
3. **Popularity** – Haskell is much more widely used and better documented

Who Uses λProlog?

There are many implementations of λProlog... but what is it used for?

1. **Theorem Proving**
    - Modelling logic frameworks
    - Appel (1999), Appel and Felty (2004), Guidi et al. (2019), Kohlhase et al. (2020)

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There are many implementations of λProlog... but what is it used for?

1. **Theorem Proving**
   - Modelling logic frameworks
   - Appel (1999), Appel and Felty (2004), Guidi et al. (2019), Kohlhase et al. (2020)

2. **Program Verification**
   - Modelling program specifications
   - Andrews (1997), Southern and Nadathur (2014)

Who Uses λProlog?

There are many implementations of λProlog... but what is it used for?

1. **Theorem Proving**
   - Modelling logic frameworks
   - Appel (1999), Appel and Felty (2004), Guidi et al. (2019), Kohlhase et al. (2020)
2. **Program Verification**
   - Modelling program specifications
   - Andrews (1997), Southern and Nadathur (2014)
3. **Algorithm Design**
   - Modelling search algorithms and compiler design
   - Rollins and Wing (1990), Liang (2002)

Alternatives to λProlog

Languages Inspired by λProlog

- Elf – implements a more expressive logic (LF) but inspired by λProlog
- Makam – a dialect of λProlog for language prototyping (very niche)

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Alternatives to λProlog

- HiLog – extends of Prolog to higher-order logic (less general than λProlog)
- Twelf – logically more general than λProlog (also based on LF)
- Interactive Theorem Provers (Isabelle, Coq, Lean) – more control over proof search

Conclusion

λProlog ...

- is a declarative logic programming language
- extends Prolog with higher-order logic and types
- can be used for theorem proving and program verification
- mostly used in research

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Further Reading

Programming with Higher-Order Logic, Miller and Nadathur (2012)
An Overview of Lambda-Prolog, Nadathur and Miller (1988)

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Intuitionistic Logic in λProlog

The formula `p → (p ∨ q) ≡ p ∨ (p → q)` is a **tautology** in classical logic

Proof: If we assume the LEM, then $p \to q \equiv \neg p \vee q$ holds.
Then

$$\begin{aligned} p \to (p \vee q) &\equiv \neg p \vee (p \vee q) \\ &\equiv p \vee (\neg p \vee q) \\ &\equiv p \vee (p \to q) \end{aligned}$$

Intuitionistic Logic in λProlog

The formula $\mathtt{p} \rightarrow (\mathtt{p} \lor \mathtt{q}) \equiv \mathtt{p} \lor (\mathtt{p} \rightarrow \mathtt{q})$ is a tautology in classical logic

Example 1

Consider trying to prove goal $\mathtt{p} \rightarrow (\mathtt{p} \lor \mathtt{q})$ from the empty program.
This is provable if and only if $\mathtt{p} \lor \mathtt{q}$ is provable from $\mathtt{p}$, which is trivial.

Example 2

Consider trying to prove goal $\mathtt{p} \lor (\mathtt{p} \rightarrow \mathtt{q})$ from the empty program.
This is provable if and only if either $\mathtt{p}$ is provable from the empty program or if $\mathtt{q}$ is provable from $\mathtt{p}$.
Neither of these is possible!

Conclusion: λProlog semantics are **unsound** for classical logic