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Interactive Theorem Proving using Isabelle/HOL

Session 3

René Thiemann

Department of Computer Science

Outline

- Natural Deduction Revisited
- Case Analysis and Structural Induction for Data Types

Natural Deduction Revisited

Last Lecture: Natural Deduction in Isabelle

- typical proof step: `from this` *more_facts* `have` *label*`:` *term* `by (rule` *thm*`)`
- three problems
  - finding names of theorems such as *thm*
  - repetitive long commands, e.g., `from this have`
  - management of labels (tedious, not informative, ...)

Use the Isabelle Library

- Isabelle already provides several theorems, e.g., inference rules of natural deduction, properties of numbers, properties of lists, ...
- to increase efficiency, these theorems should be re-used, not re-proved
- problem: how to know the name of all these theorems, e.g.,

```
thm excluded_middle disjI1 exE ccontr
thm add.commute add_le_cancel_right
```

- solution: use search engine to quickly find
  - already proven theorems
  - already defined constants, e.g., algorithms on lists, numbers, sets, ...

Finding Existing Theorems

- enter query in “Query/Find Theorems” panel or after `find_theorems` command
- scope: search is restricted to accessible content in current theory, including imports

Search Criteria

- `name: foo` – search for facts whose name contains substring “`foo`”
- `"`*pattern*`"` – search for facts that match *pattern*
- prefix criterion by “`-`” to exclude facts that match
- combine several criteria by juxtaposition

Search Patterns

HOL terms with schematic variables `?x`, `?y`, ... or `_` instead of free variables

Examples

query	finds facts mentioning	query	finds facts mentioning
`"_ + _"`	addition	`2 "(+)"`	2 and addition function
`"?x + ?x"`	addition of same value	`"_ * (_ + _) = _"`	distributive law

Finding Existing Constants

- enter query in “Query/Find Constants” panel or after `find_consts` command
- scope: search is restricted to accessible content in current theory, including imports

Search Criteria

- `name: foo` – search for constants whose name contains substring “`foo`”
- `"`*type*`"` – search for constants that match a specific *type*
- combine several criteria by juxtaposition

Search Types

HOL types with schematic type variables `?'a`, `?'b`, ... or _ instead of free type variables

Example

```
find_consts "?'a ⇒ ?'a ⇒ _ list" name: "List"
```

searches for all binary functions where first and second argument have the same type, that return a list, and whose names includes `"List"` (e.g., as theory-prefix of a long name)

Abbreviations of Statements

- `then` = `from this`
  (unlike to `from`, after `then` no further facts may be stated)
- `hence` = `then have`
- `thus` = `then show`
- `with` *facts* = `from` *facts* `this`

Passing Auxiliary Facts

- instead of passing facts before the property to be proven, one can also state facts after the property via `using`:

  `from` *facts* `have` *proposition* ⟨*proof*⟩

  is equivalent to

  `have` *proposition* `using` *facts* ⟨*proof*⟩

  - style: state important facts before, and auxiliary facts after *proposition*
  - caution: label `this` is not available after `using`

Avoiding Labels: `moreover` and `ultimately`

- often proofs are of the form that auxiliary properties 1, . . . , n are proven and then one can conclude
- manually labeling all these properties is tedious, in particular if labels are somehow sorted and one needs to insert something in the middle
- use `moreover` and `ultimately` to write these proofs without explicit labels
- example

with labels

```
have 1: A ⟨proof⟩
have B ⟨proof⟩
hence 2: C ⟨proof⟩
have D ⟨proof⟩
hence 3: E ⟨proof⟩
from 1 2 3 show ?thesis
```

without labels

```
have A ⟨proof⟩
moreover                              (* store A *)
have B ⟨proof⟩
hence C ⟨proof⟩
moreover                              (* store C *)
have D ⟨proof⟩
hence E ⟨proof⟩
ultimately show ?thesis (* A C E are avail. *)
```

Case Analysis on Booleans

- Isabelle provides special syntax to perform proofs by case analysis
- this slide: case analysis on Booleans (general case: later)
- structure is as follows, where *term* is of type `bool` (copy outline from output panel)

```
proof (cases term)   (* here outline is displayed in output panel *)
  case True
  ...   (* label True refers to fact "term" *)
  show ?thesis ⟨proof⟩
next
  case ownLabel: False
  ...   (* label ownLabel refers to fact "~ term" *)
  show ?thesis ⟨proof⟩
qed
```

- order of cases is irrelevant, separation of cases via `next`
- user-defined labels become important in nested case analyses
- omitted case(s) can be solved via final method, e.g., `qed auto`

The `rule` Method – Revisited

- `rule` *fact* – if provided facts are empty, apply *fact* as introduction rule (last week)
- otherwise, apply *fact* as elimination rule
- introduction rule: conclusion introduces connective $(\ldots \Longrightarrow \mathtt{A} \wedge \mathtt{B})$
- elimination rule: premise contains connective that is eliminated $(\mathtt{A} \wedge \mathtt{B} \Longrightarrow \ldots)$

Rule Application

- given rule $\mathtt{P}_1 \Longrightarrow \ldots \Longrightarrow \mathtt{P}_n \Longrightarrow \mathtt{C}$
- intro – unify $\mathtt{C}$ with conclusion of current subgoal and add correspondingly instantiated premises $\mathtt{P}_1\sigma, \ldots, \mathtt{P}_n\sigma$ as new subgoals
- elim – unify major premise $\mathtt{P}_1$ of rule with first of current facts; unify remaining current facts with remaining premises; add rest of premises correspondingly instantiated as new subgoals

Beyond rule – intro and elim

- the `rule` method applies exactly one rule (intro or elim)
- the `intro` method applies several introduction rules exhaustively
- the `elim` method applies several elimination rules exhaustively

Example

```
lemma "A ∧ (∃ x :: nat. B x ∧ (C ∨ D x))"
proof (intro conjI exI)
  — ‹three subgoals: A, B ?x, C ∨ D ?x›
  show A ⟨proof⟩
  show "B 5" ⟨proof⟩      (* here we choose witness 5 *)
  show "C ∨ D 5" ⟨proof⟩ (* no choice of witness anymore *)
qed
```

Case Analysis and Structural Induction for Data Types

Data Type Definitions

- whenever a data type `ty` is defined, in the background several theorems are proven
  - they can be inspected via `print_theorems` directly after the definition
  - simplification rules: `ty.simps` (automatically used by `auto`)
  - case analysis rule: `ty.exhaust` (used by `cases "`*term*` :: ty"`)
  - induction rule: `ty.induct` (used by `induction "`*variable*` :: ty"`)

Example

- consider Isabelle's lists: `datatype 'a list = Nil | Cons 'a "'a list"`
- special syntax: `[]` is the same as `Nil`, `#` is an infix operator for `Cons`, and there is syntax such as `[x, y, z]`
- `list.simps` contains among others `(x # xs = y # ys) = (x = y ∧ xs = ys)`
  `(case x # xs of [] ⇒ e | y # ys ⇒ f y ys) = f x xs`
- `list.exhaust:` $(\texttt{ys = []} \Longrightarrow \texttt{P}) \Longrightarrow (\bigwedge \texttt{x xs. ys = x \# xs} \Longrightarrow \texttt{P}) \Longrightarrow \texttt{P}$
- `list.induct:` $\texttt{P []} \Longrightarrow (\bigwedge \texttt{x xs. P xs} \Longrightarrow \texttt{P (x \# xs)}) \Longrightarrow \texttt{P ys}$

Function Definitions

- whenever a function `f` is defined, in the background several theorems are proven
  - they can be inspected via `print_theorems` directly after the definition
  - simplification rules: `f.simps` (automatically used by `auto`)
  - induction rule: `f.induct` (details in upcoming lecture)

Example

- consider append function:

```
fun app :: "'a list ⇒ 'a list ⇒ 'a list" where
  "app [] ys = ys"
| "app (x # xs) ys = x # (app xs ys)"
```

- `app.simps` are the two defining equations as theorems

The induction Method

- `induction x` – induction on parameter `x` (rule chosen according to type of `x`)
- use `case` to start case
  - syntax: `case` (*CName* x_1 ... x_n) where
    - *CName* is name of constructor
    - $x_1, \ldots, x_n$ are freely chosen variable names that represent the arguments of *CName*
  - *CName* is also label that contains the IHs;
    e.g., for binary tree with constructor `Node`, the fact `Node(1)` would be the first IH (left subtree) and `Node(2)` would be the second IH (right subtree)
- `?case` abbreviates goal of current case, separate cases by `next`
- outline of induction proof is available in output panel for `induction x` method

The cases Method

- `cases` *term* – case analysis on parameter *term* (rule chosen according to type of *term*)
- same structure as induction method, with two differences
  - goals of current case are still `?thesis`, not `?case`
  - no IHs are available as facts, but equalities *term* = *CName* x_1 ... x_n

Demo – List Reversal

```
fun app :: "'a list ⇒ 'a list ⇒ 'a list" where
  "app [] ys = ys"
| "app (x # xs) ys = x # (app xs ys)"

fun reverse :: "'a list ⇒ 'a list" where
  "reverse [] = []"
| "reverse (x # xs) = app (reverse xs) ([x])"

lemma rev_rev: "reverse (reverse xs) = xs"
```

Proof Strategies

1. perform induction on suitable variable (more on that next week)
2. copy proof outline by click in blue part of output panel; adjust variable names on demand
3. handle each case, replace `sorry` by `proof auto`
   - if successful, replace `proof auto` by `by auto`
   - if not, either
     - perform proof manually (natural deduction, add intermediate statements, ...)
     - or identify required lemma to make progress and first prove that lemma
4. cleanup proof, e.g., drop trivial cases and replace final `qed` by `qed auto`

Auxiliary Lemmas

- currently: assume auxiliary lemmas are just equations $lhs = rhs$
- formulate lemmas such that lhs is larger than rhs, so that terms get smaller
- activate lemma globally via `[simp]`-attribute: `lemma useful[simp]: "`$lhs = rhs$`"`
- activate lemmas locally: `proof (auto simp: useful ...)`
- warning: if the activated equations do not terminate, then `auto` might not terminate