



Interactive Theorem Proving using Isabelle/HOL

Session 6

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Outline

- Projects
- Proof Methods
- Sledgehammer

Projects

Projects

- 1–3 person projects
- for many person projects individual contributions have to be clarified
- all projects can be started quite soonish
(lacking knowledge for some projects: **inductive** definitions and **sets**)
- evaluation rules: [website](#)
- project topics (details: [website](#))
 - Congruence Closure (2–3 persons)
 - Pattern-Completeness (2–3 persons)
 - A Compiler for the Register Machine from Hell (2 persons)
 - Propositional Logic (2 persons)
 - Tseitin Transformation (2 persons)
 - BIGNAT - Natural Numbers of Arbitrary Size (1 person)
 - The Euclidean Algorithm - Inductively (1 person)
- project assignment after break

Proof Methods

Last Session: Attributes

- attributes can **modify facts**: of, OF, symmetric, rule_format, simplified, ...
- attributes can also specify **usage of facts**; examples
 - **how** to declare that rule should be used in specific method, e.g., simplification
 - `lemma fact[simp]: ...` when stating lemma
 - `declare fact[simp]` outside proof
 - `note [simp] = fact` locally within proof
 - **what** to declare
 - `declare fact[simp]` add to standard simpset
 - `declare fact[simp del]` delete from standard simpset
 - `declare fact[termination_simp]` add to termination simpset
 - `declare fact[intro]` declare as **introduction rule**
 - `declare fact[elim]` declare as **elimination rule**
 - `declare fact[dest]` declare as **destruction rule**

Kinds of Rules

- simplification rules – (conditional) equations used from left to right
- introduction rules – if conclusion of rule matches conclusion of subgoal, replace it by premises of rule (generating one new subgoal per premise)
- destruction rules – replace first premise of subgoal matching major premise of rule by conclusion (together with remaining premises) of rule
- elimination rules – like destruction rules, but rule is supposed to not lose (destruct) information (compare `conjunct1` with `conjE`)

Examples

- `have "∀x. P x" apply (rule allI) ↪ $\bigwedge x. P x$`
- `have "A ∧ B ⇒ C" apply (drule conjunct2) ↪ $B ⇒ C$`
- `have "A ∨ B ⇒ C" apply (erule disjE) ↪`

$$\begin{array}{l} 1. A ⇒ C \\ 2. B ⇒ C \end{array}$$

(`drule` and `erule` are designed to apply dest-rules and elim-rules, respectively)

Equational Proof Methods

- `unfold fact+` – exhaustively apply equational facts (replacing left-hand sides by right-hand sides); usually as initial method
- `simp/simp_all` – exhaustively apply simp rules to first/all subgoal(s)

Proof Methods for Classical Reasoning

- `(intro | elim) fact+` – exhaustively apply intro/elim rules; usually as initial method
- `blast (best, fast)` – solve first subgoal by exhaustive proof search (up to certain bound) using **all known intro/dest/elim rules** (using best-first search, depth-first search)

Combined Proof Methods

- `force (fastforce, bestsimp)` – **solve first subgoal** by combination of equational and classical reasoning
- `auto` – **apply** combination of equational and classical reasoning to all subgoals and leave result as new subgoals

Selection of Methods

- distinction between
 - **initial methods** (predictable outcome, used at start of proof, e.g. rule, intro, dest, unfold, ...)
 - **final methods** (solve some proof goals, e.g., fast, best, auto, blast, linarith, presburger, algebra, metis, smt, ...)
- problem: how to know all the methods?
- solution
 - learn initial methods
 - use **try0** to find suitable final method, it will try out several known methods and then inform about success
 - example

```
lemma "∀x. ∃y. P x y ⇒ ∃f. ∀x. P x (f x)"
  try0
```

(* output window shows successful method, e.g., by metis;
after insertion of method, try0-invocation should be eliminated *)

Modifiers of Methods

success of methods can be increased by manual adaptations, e.g., addition of simp rules

Modifiers for Classical Methods

classical methods (like `blast` and `auto`) take following modifiers:

- `intro: fact+` – add additional intro rules
- `dest: fact+` – add additional dest rules
- `elim: fact+` – add additional elim rules
- `del: fact+` – delete classical rules

Note

when used with combined methods (like `force` and `auto`), modifiers for simplifier use prefix `simp` (like `simp add:`, `simp del:`, ...)

The Split-Modifier

- consider goal that requires a case-analysis because of a case-expression, e.g. on lists sorted (`case g x of [] => [5] | y # ys => ys @ zs @ [y]`)
- for each datatype **split rules** are created that support such a case-analysis (`nat.splits`, `prod.splits`, `list.splits`, `bool.splits`, ...)
- **split rules are equalities** that can be used by the simplifier, e.g., for lists:

$$P (\text{case } xs \text{ of } [] \Rightarrow c \mid y \# ys \Rightarrow f \ y \ ys) = \\ ((xs = [] \longrightarrow P \ c) \wedge (\forall y \ ys. xs = y \# ys \longrightarrow P \ (f \ y \ ys)))$$

- split rules have to be activated manually via `split-modifier`, syntax is `split: fact+`
- `split-modifier` works in methods that use the simplifier: `simp`, `auto`, `force`, ...
- example

```
have "sorted (case g x of [] => [5] | y # ys => ys @ zs @ [y])"
apply (simp only: split: list.splits)
```

```
1. (g x = [] → sorted [5]) ∧
   (∀y ys. g x = y # ys → sorted (ys @ zs @ [y]))
```

remark: `only`-modifier changes `simpset` so that only specified facts are used (here: none)

Demo

soundness of mergesort via modifiers

Composition of Methods

- sometimes, it is useful to apply several methods sequentially, e.g.,

```
lemma "∀ x :: nat. x < 30 → (∃ y z. y + x ≤ z ∧ odd y ∧ odd z)"
  apply (intro allI impI)
  apply (rule exI[of _ 5])
  apply (rule exI[of _ 35])
  by auto
```

- instead of using several `apply`s, one can combine methods sequentially via `,` or `;`

```
lemma "∀ x :: nat. x < 30 → (∃ y z. y + x ≤ z ∧ odd y ∧ odd z)"
  by (intro allI impI, rule exI[of _ 5], rule exI[of _ 35], auto)
```

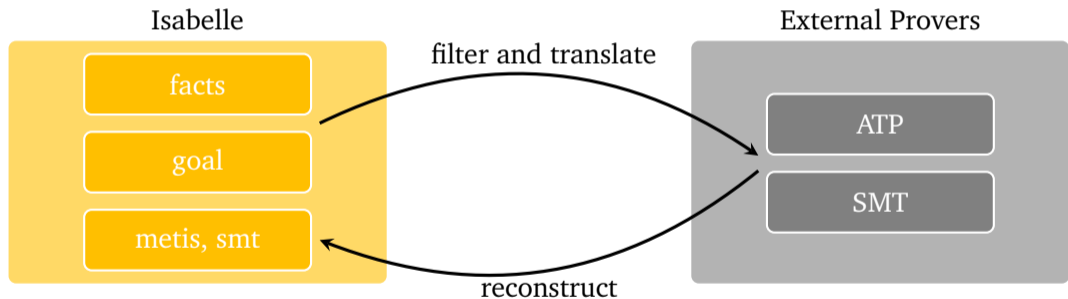
- `apply (method1, method2)` is the same as `apply method1 apply method2`
- `apply (method1; method2)` first apply `method1` and apply `method2` on **all new subgoals** that are produced by `method1`

- (dis)advantages of sequential composition of methods
 - + fast to type; supports nested cases, e.g., `by (cases xs; cases ys; auto)` triggers case-analysis on all four combinations of whether lists `xs` and `ys` are (non)empty
 - excessive use is hard to maintain and read, since no intermediate proof goals are visible

Sledgehammer

Sledgehammer

tool that applies automated theorem provers (ATPs) and satisfiability-modulo-theory (SMT) solvers to current subgoal



Phase 1: From Isabelle to External Provers

aim: prove $\Phi \models \psi$ where Φ is collection of all available facts and ψ is current goal

- **selection problem**
 - find-theorems after loading Main shows 22200 theorems ($\leq |\Phi|$)
 - current ATPs are not performing well when using all available facts
 - **relevance filter**: select top N facts that might be relevant for current goal
 - choice of N depends on target ATP
 - different relevance filters available, e.g., syntax guided or trained via machine learning
- **language problem**
 - untyped FOL (ATP) \neq typed HOL (Isabelle) \neq SMT languages
 - solution: encoding (e.g., encode type-information into terms, etc.)
 - adds a certain amount of imprecision
- **overall workflow**: for each external prover P (in parallel)
 - select $\{\varphi_1, \dots, \varphi_{N_p}\} \subseteq \Phi$ by relevance filter
 - ask P to prove $encode_P(\varphi_1 \longrightarrow \dots \longrightarrow \varphi_{N_p} \longrightarrow \psi)$
 - collect successful proofs

Phase 2: From External Provers to Isabelle

aim: prove $\Phi \models \psi$ where Φ is collection of all available fact and ψ is current goal

phase 1: obtain proof of $encode_P(\varphi_1 \longrightarrow \dots \longrightarrow \varphi_{N_P} \longrightarrow \psi)$

- **reconstruction problem**
 - external proof is unreliable (buggy external provers)
 - external proof is non-trivial to replay in Isabelle (e.g., imprecision of encoding)
 - solution
 - analyze external proof: **which φ_i have been used when proving ψ ?**
 - reconstruction of proof by finding **HOL-proof using Isabelle inferences**, where search is started from scratch, but restricted to used φ_i
- **metis**
 - metis is Isabelle built-in ATP (first-order ordered resolution and paramodulation)
 - its inferences go through Isabelle's proof kernel (correct by construction)
 - **metis fact*** – apply metis using some auxiliary facts, e.g., the used φ_i 's
- **smt**
 - alternative reconstruction mechanism to metis
 - main conceptual difference: for finding suitable inferences, again SMT solvers are invoked

Sledgehammer in Action

- standalone: via command `sledgehammer` (available in proof-mode)
 - `have statement sledgehammer` or `apply method sledgehammer` but not `have statement proof simp sledgehammer`
 - after invocation wait some seconds on answer in output panel (or abort by erasing `sledgehammer` command)
 - copy successful proof from output panel; erase `sledgehammer` command
- in combination: `try` combines `try0` with `sledgehammer`
 - note: in `try`, `sledgehammer` has a rather short time-limit, unlike in standalone version
- separate user manual for sledgehammer is available: `isabelle doc sledgehammer`

Strategies for Sledgehammer and Find-Theorems

- sledgehammer is only applicable if it completely solves a goal (all or nothing)
 - strategy: if sledgehammer cannot solve a goal in one step, add intermediate goals manually
- find-theorems helps you more in exploring possibilities and getting names
 - what kind of theorems are there to prove $\sum \dots = \sum \dots$?
 - what is the name of the distributivity law between addition and multiplication?

Demo

$\sqrt{2}$ is irrational