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Interactive Theorem Proving using Isabelle/HOL

Session 10

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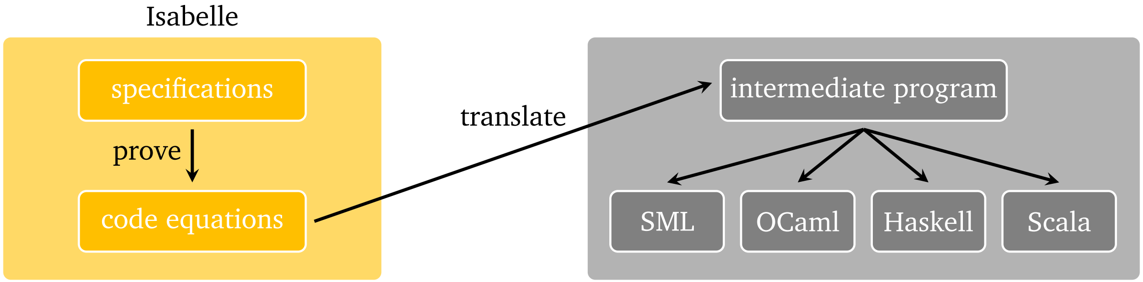
Outline

- Code Generation
- Code Equations Beyond Defining Equations
- Conditional Code Equations

Code Generation

Code Generator Architecture

- code equations – executable subset of Isabelle/HOL specifications of shape
  `f` t_1 `...` t_n `= ...`
- code equations are translated into intermediate program with datatypes and functions
- intermediate program is serialized into concrete programming language



Note

pen-and-paper proof that translation guarantees partial correctness [1]

Usage of the Code Generator

- `value (code) "sort [7, 4, 8 :: nat]"` – evaluate some expression
- `lemma "sort [7, 4 :: nat] = [4, 7]" by code_simp` – proof by evaluation
- `lemma "sort [7, 4 :: nat] = [4, 7]" by eval` – proof by evaluation
- `lemma "sorted [x,y]" quickcheck` – find counterexample by instantiation and evaluation
- `export_code sort in` *Language* – generate code for `sort` in *Language*

remark: `code_simp` and `eval` differ

- `code_simp` – code equations are applied via Isabelle kernel (trusted)
- `eval` – reflection mechanism: code equations are translated to SML, compiled on the fly, then SML evaluation is started, and SML result `true` is reflected as Isabelle result `True` (more efficient)

Exporting Haskell Code

- `code_thms f` – print code equations for `f`
- `export_code f g in Haskell` – generate Haskell code for functions `f` and `g`
- `export_code f in Haskell module_name Name` – generate code as module `Name`

Demo – Reverse

```
fun rev :: "'a list ⇒ 'a list" where
  "rev [] = []"
| "rev (x # xs) = rev xs @ [x]"
code_thms rev
export_code rev in Haskell module_name Rev1
```

- append equations are visible in `code_thms`
- however, Isabelle's append is mapped to Haskell's append function `(++)`
- similarly, Isabelle's list type is mapped to Haskell's list type
- mapping of Isabelle constants/types to target language const./types won't be discussed

Declaring Code Equations

- some commands, like `fun` and `definition`, implicitly declare code equations
- `declare` *fact* `[code del]` – remove code equation *fact*
- `declare [[code drop:` f `... ]]` – remove all code equations for functions f ...
- use attribute `[code]` to declare code equation

Demo – Efficient Code of Reverse Function (Program Refinement)

```
fun itrev :: "'a list ⇒ 'a list ⇒ 'a list" where
  "itrev [] acc = acc"
| "itrev (x # xs) acc = itrev xs (x # acc)"

lemma itrev_rev [simp]: "itrev xs ys = rev xs @ ys" ⟨proof⟩
declare [[code drop: rev]] (* drop old implementation of rev *)
lemma rev_code [code]: "rev xs = itrev xs []" ⟨proof⟩
code_thms rev (* obtain improved (refined) code equations *)
```

Code Unfold – Automatic Rewriting of Code Equations

- some functions are not executable, in particular if defining equations contain quantifiers
  `definition "test0 = (∀ x :: nat. even x)"`
- however, certain patterns with quantifiers look executable
  `definition "test1 (xs :: nat list) = (Ball (set xs) even)"`
  reason: bounded quantification over `set xs` is identical to iteration over all list elements
- such an implementation for bounded quantification can be expressed via an equation
  `lemma [code_unfold]: "Ball (set xs) p = list_all p xs"` ⟨*proof*⟩
- effect of code unfold lemma
  - whenever rhs of code equation contains pattern `Ball (set xs) p` then it will be rewritten to `list_all p xs`
  - in example: code equation for `test1` gets rewritten to
    `test1 xs = list_all even xs`

Code Equations might Introduce Type-Class Constraints

- some functions are not executable in their original form
- code equations can introduce additional type-class constraints
- example

  ```
  definition test2 :: "('a ⇒ bool) ⇒ bool" where
    "test2 p = (∃ x. p x)"
  ```

  Isabelle generates code for `test2` with the additional restriction that `'a` must be a type in the `enum`-class, i.e., all elements of that type must be enumerable via a list
- consequences
  - `definition "test2_nat = test2 (λ x :: nat. x > 5)"` – code generation fails
  - `definition "test2_char = test2 (λ x. x > CHR ''a'')"` – code generation succeeds

Code Equations Beyond Defining Equations

Code Equations – Limits and Opportunities

- limit: via code generation we will only get partial correctness

  if evaluation of generated code on input returns some result, then this result is correct
- opportunity: code equations can be arbitrary equations that can be proven
- examples
  - program refinement (write more efficient code equations):
    `lemma [code]: "rev xs = itrev xs []"`
  - implement any function in a trivial way: `lemma [code]: "f x y = f x y"`
- upcoming: examples illustrating the power of code equations

Code Equations – Partial Implementations

```
definition "complex_predicate (x :: nat) = (x > 894105890)"
(* assume we don't know the rhs, might be complex algorithm *)

definition "unknown_problem = (∃ x. complex_predicate x)"
(* unknown problem is not executable *)

lemma [code]: "unknown_problem = (
  if (∃ x ∈ set [0..<10000]. complex_predicate x) then True
  else unknown_problem)" ⟨proof⟩
(* unknown problem will be executable and loops *)

lemma [code]: "unknown_problem = (
  if (∃ x ∈ set [0..<10000]. complex_predicate x) then True
  else Code.abort (STR ''giving up'') (λ _. unknown_problem))" ⟨proof⟩
(* unknown problem will be executable and fails *)
(* "Code.abort e (% _ . x) = x" in logic; throws error in evaluation *)
```

Code Equations – Phantom Arguments

we can implement Isabelle functions by functions that have auxiliary arguments that just exist in the implementation

```
definition approx_problem :: "nat ⇒ bool" where
  "approx_problem n = unknown_problem"
(* n is phantom argument *)

lemma [code]: "approx_problem n = (if complex_predicate n then True
    else approx_problem (n + 1))" ⟨proof⟩
(* n controls the implementation *)

lemma [code]: "unknown_problem = approx_problem 0" ⟨proof⟩

lemma unknown_problem by eval
(* evaluation succeeds because of unbounded iteration *)
```

Approximation Algorithm without Termination Proof

```
definition property :: "real ⇒ bool" ...
definition approx :: "nat ⇒ real ⇒ rat × rat" ...
(* approximate real with precision n, e.g., via lower and upper bound *)

definition approx_alg :: "rat × rat ⇒ bool option" ...
lemma "approx n r = lu ⟹ approx_alg lu = Some b ⟹ b = property r"
(* if approximation is successful, then property is determined *)

definition check_property :: "nat ⇒ real ⇒ bool" where
  "check_property n r = property r"   (* impl. with phantom argument *)

lemma [code]: "check_property n r =
  (case approx_alg (approx n r) of
     Some b ⇒ b
   | None ⇒ check_property (n+2) r)" (* increase precision by 2 *)

lemma [code]: "property r = check_property 10 r"
```

Conditional Code Equations

Reachability in Graphs – Conditional Code Equations

```
context
  fixes G :: "'a rel"     (* fix local parameters (here: a graph) *)
  assumes fG: "finite G"  (* add assumptions (here: graph is finite) *)
begin (* context with G *)
fun reach_main :: "'a set ⇒ 'a set ⇒ 'a set" where
  "reach_main todo reached = (if todo = {} then reached
     else let successors = snd ` (Set.filter (λ (x,y). x ∈ todo) G);
                  new = successors - reached
              in reach_main new (reached ∪ successors))"
(* termination proof is not automatic, and requires finiteness of G! *)

definition "reach A = reach_main A A"
lemma "reach A = {y. ∃ x ∈ A. (x,y) ∈ G^*}" ⟨proof⟩
end (* of context *)

thm reach_main.simps (* outside context obtain conditional equation *)
(* finite G ==> reach_main G todo reached = (if todo = ... ) *)
```

Conditional Code Equations

- problem: conditional code equations `cond x` $\Longrightarrow$ `lhs x = rhs x` are not accepted by code generator: code equations must be unconditional!
- possible solutions
  1. move condition into code equation
     `lhs x = (if cond x then rhs x else (Code.abort) (lhs x))`
     disadvantage: condition is checked in every iteration
  2. create dedicated type `typedef 'a ctyp = { x :: 'a. cond x }`,
     check condition initially once, but not in every iteration,
     work with lift-definitions to convert between types
  3. if the conditional code equations are tail-recursive, use `partial_function` to define equivalent unconditional code equations, avoids type-conversions
  4. just define desired property and from that prove a code equation without explicit function definition
- all solutions will be illustrated via the reachability example

Solution 1 – Move Condition into If-Then-Else

```
definition "err = STR ''reach invoked on infinite graph''"

lemma [code]:
  "reach_main G todo reached = (if finite G (* check cond *) then
     if todo = {} then reached
     else let successors = snd ` (Set.filter (λ (x,y). x ∈ todo) G);
              new = successors - reached
          in reach_main G new (reached ∪ successors)
     else Code.abort err (λ _. reach_main G todo reached))" ⟨proof⟩

lemma [code]: "reach G A = (if finite G then reach_main G A A
     else Code.abort err (λ _. reach G A))" ⟨proof⟩

value (code) "reach {(1,2 :: nat), (3,4), (2,4), (4,1)} {1}"
(* {4, 2, 1} *)
```

Solution 2 – Create Type for Condition

```
typedef 'a fset = "{ A :: 'a set. finite A}" by auto
setup_lifting type_definition_fset

lift_definition get_set :: "'a fset ⇒ 'a set" is "λ A. A" .

lemma "finite (get_set A)" ⟨proof⟩

definition "reach_main_2 fG = reach_main (get_set fG)"

lemma [code]: "reach_main_2 fG todo reached = (if todo = {}
  then reached else let
    successors = snd ` (Set.filter (λ (x,y). x ∈ todo) (get_set fG));
    new = successors - reached
  in reach_main_2 fG new (reached ∪ successors))" ⟨proof⟩
```

Solution 2 – Continued

```
definition "reach_2 fG = reach (get_set fG)"

lemma [code]: "reach_2 fG A = reach_main_2 fG A A" ⟨proof⟩

(* problems: create elements of fset; get connection to reach *)

lift_definition (code_dt) get_fset :: "'a set ⇒ 'a fset option" is
  "λ G. if finite G then Some G else None" ⟨proof⟩

lemma [code]: "reach G A = (case get_fset G of
    Some fG ⇒ reach_2 fG A
  | None ⇒ Code.abort err (λ _. reach G A))" ⟨proof⟩

(* note: (code_dt) is required to obtain executable code,
   since lifted type (fset) is wrapped within other type (option) *)
```

Solution 3 – `partial_function`

- `partial_function (tailrec)` allows user to specify unconditional defining equation, even if they are non-terminating, provided that the equation is tail-recursive
- syntactic constraints are similar to `definition`, except that recursion is allowed
- logically, non-termination is modeled via `undefined`

```
partial_function (tailrec) (* copy of reach_main *)
  reach_main_3 :: "'a rel ⇒ 'a set ⇒ 'a set ⇒ 'a set" where
  [code]: "reach_main_3 G todo reached = (if todo = {} then reached
     else let successors = snd ` (Set.filter (λ (x,y). x ∈ todo) G);
              new = successors - reached
          in reach_main_3 G new (reached ∪ successors))"
definition "reach_3 G A = reach_main_3 G A A" (* copy of reach *)

lemma "finite G ⟹ reach_3 G A = reach G A" (* via reach_main.induct *)
lemma [code]: "reach G A = (if finite G then reach_3 G A
    else Code.abort err (λ _. reach G A))" ⟨proof⟩
```

Solution 4 – No Specification of Algorithm, Just Code Equation

```
definition reach' :: "'a rel ⇒ 'a set ⇒ 'a set" where
  "reach' G A = {y. ∃x∈A. (x, y) ∈ G^*}"

lemma [code]: "reach' G A = (if A = {} then {} else
   let A_edges = Set.filter (λ (x,y). x ∈ A) G;
       successors = snd ` A_edges
       in A ∪ reach' (G - A_edges) successors)" ⟨proof⟩

value (code) "reach' {(1,2 :: nat), (3,4), (2,4), (4,1)} {1}"
(* {2, 4, 1} *)
```

Further Reading

Florian Haftmann and Tobias Nipkow.
Code generation via higher-order rewrite systems.
In *FLOPS*, volume 6009 of *LNCS*, pages 103–117. Springer, 2010.
`doi:10.1007/978-3-642-12251-4_9`.

Florian Haftmann and Lukas Bulwahn.
Code generation from Isabelle/HOL theories.
isabelle doc codegen, 2021.