universität innsbruck

SS 2023

Program Verification

Part 6 – Verification of Imperative Programs

René Thiemann

Department of Computer Science

Imperative Programs

Imperative Programs

- we here consider a small imperative programming language
- it consists of
  - arithmetic expressions $\mathcal{A}$ over some set of variables $\mathcal{V}$

$$\frac{n \in \mathbb{Z}}{n \in \mathcal{A}} \qquad \frac{x \in \mathcal{V}}{x \in \mathcal{A}} \qquad \frac{\{e_1, e_2\} \subseteq \mathcal{A} \quad \odot \in \{\texttt{+},\texttt{-},\texttt{*}\}}{e_1 \odot e_2 \in \mathcal{A}}$$

  - Boolean expressions $\mathcal{B}$

$$\frac{c \in \{\texttt{true}, \texttt{false}\}}{c \in \mathcal{B}} \qquad \frac{\{e_1, e_2\} \subseteq \mathcal{A} \quad \odot \in \{\texttt{=},\texttt{<},\texttt{<=},\texttt{!=}\}}{e_1 \odot e_2 \in \mathcal{B}}$$

$$\frac{b \in \mathcal{B}}{\texttt{!}b \in \mathcal{B}} \qquad \frac{\{b_1, b_2\} \subseteq \mathcal{B} \quad \odot \in \{\texttt{\&\&},\texttt{||}\}}{b_1 \odot b_2 \in \mathcal{B}}$$

  - commands $\mathcal{C}$

Commands and Programs

- commands $\mathcal{C}$ consist of
  - assignments
    $$\frac{x \in \mathcal{V} \quad e \in \mathcal{A}}{x := e \in \mathcal{C}}$$
  - if-then-else
    $$\frac{b \in \mathcal{B} \quad \{C_1, C_2\} \subseteq \mathcal{C}}{\texttt{if}\ b\ \texttt{then}\ C_1\ \texttt{else}\ C_2 \in \mathcal{C}}$$
  - sequential execution
    $$\frac{\{C_1, C_2\} \subseteq \mathcal{C}}{C_1\,;C_2 \in \mathcal{C}}$$
  - while-loops
    $$\frac{b \in \mathcal{B} \quad C \in \mathcal{C}}{\texttt{while}\ b\ \{C\} \in \mathcal{C}}$$
  - no-operation
    $$\frac{}{\texttt{skip} \in \mathcal{C}}$$
  - curly braces are added for disambiguation, e.g. consider
    `while x < 5 { x := x + 2 } ; y := y - 1`
- a program P is just a command C

Verification

- partial correctness predicate via Hoare-triples: $\models (\!|\varphi|\!)\, P\, (\!|\psi|\!)$
  - semantic notion
  - meaning: whenever initial state satisfies φ,
  - and execution of P terminates,
  - then final state satisfies ψ
  - φ is called precondition, ψ is postcondition
  - here, formulas may range over program variables and logical variables
  - clearly, $\models$ requires semantic of commands
- Hoare calculus: $\vdash (\!|\varphi|\!)\, P\, (\!|\psi|\!)$
  - syntactic calculus (similar to natural deduction)
  - sound: whenever $\vdash (\!|\varphi|\!)\, P\, (\!|\psi|\!)$ then $\models (\!|\varphi|\!)\, P\, (\!|\psi|\!)$

Semantics – Expressions

- state is evaluation $\alpha : \mathcal{V} \to \mathbb{Z}$
- semantics of arithmetic and Boolean expressions are defined as
  - $[\![\cdot]\!]_\alpha : \mathcal{A} \to \mathbb{Z}$
    e.g., if $\alpha(x) = 5$ then $[\![6 * x + 1]\!]_\alpha = 31$
  - $[\![\cdot]\!]_\alpha : \mathcal{B} \to \{\mathsf{true}, \mathsf{false}\}$
    e.g., if $\alpha(x) = 5$ then $[\![6 * x + 1 < 20]\!]_\alpha = \mathsf{false}$
- we omit the straight-forward recursive definitions of $[\![\cdot]\!]_\alpha$ here

Semantics – Commands

- semantics of commands is given via small-step-semantics defined as relation $\hookrightarrow \subseteq (\mathcal{C} \times (\mathcal{V} \to \mathbb{Z}))^2$

$$\frac{}{(x := e, \alpha) \hookrightarrow (\mathtt{skip}, \alpha[x := [\![e]\!]_\alpha])}$$

$$\frac{[\![b]\!]_\alpha = \mathsf{true}}{(\mathtt{if}\ b\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2, \alpha) \hookrightarrow (C_1, \alpha)}$$

$$\frac{[\![b]\!]_\alpha = \mathsf{false}}{(\mathtt{if}\ b\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2, \alpha) \hookrightarrow (C_2, \alpha)}$$

$$\frac{(C_1, \alpha) \hookrightarrow (C_1', \beta)}{(C_1; C_2, \alpha) \hookrightarrow (C_1'; C_2, \beta)} \qquad \frac{}{(\mathtt{skip}; C, \alpha) \hookrightarrow (C, \alpha)}$$

$$\frac{[\![b]\!]_\alpha = \mathsf{true}}{(\mathtt{while}\ b\ C, \alpha) \hookrightarrow (C; \mathtt{while}\ b\ C, \alpha)}$$

$$\frac{[\![b]\!]_\alpha = \mathsf{false}}{(\mathtt{while}\ b\ C, \alpha) \hookrightarrow (\mathtt{skip}, \alpha)}$$

- $(\mathtt{skip}, \alpha)$ is normal form

Semantics – Programs

- we can formally define $\models (\!|\varphi|\!)\ P\ (\!|\psi|\!)$ as

$$\forall \alpha, \beta.\ \alpha \models \varphi \longrightarrow (P, \alpha) \hookrightarrow^* (\texttt{skip}, \beta) \longrightarrow \beta \models \psi$$

- example specification: $(\!|x > 0|\!)\ P\ (\!|y \cdot y < x|\!)$
  - if initially $x > 0$, after running the program P, the final values of x and y must satisfy $y \cdot y < x$
  - nothing is required if initially $x \leq 0$
  - nothing is required if program does not terminate
  - specification is satisfied by program P defined as

```
y := 0
```

  - specification is satisfied by program P defined as

```
y := 0;
while (y * y < x) {
  y := y + 1
};
y := y - 1
```

Program Variables and Logical Variables

- consider program Fact

```
y := 1;
while (x != 0) {
  y := y * x;
  x := x - 1
}
```

- specification for factorial: does $\models (\!|x \geq 0|\!)\ \mathit{Fact}\ (\!|y = x!|\!)$ hold?
  - if $\alpha(x) = 6$ and $(\mathit{Fact}, \alpha) \hookrightarrow^* (\mathtt{skip}, \beta)$ then $\beta(y) = 720 = 6!$
  - problem: $\beta(x) = 0$, so $y = x!$ does not hold for final values
  - hence $\not\models (\!|x \geq 0|\!)\ \mathit{Fact}\ (\!|y = x!|\!)$, since specification is wrong
- solution: store initial values in logical variables
- in example: introduce logical variable x_0

$$\models (\!|x = x_0 \wedge x \geq 0|\!)\ \mathit{Fact}\ (\!|y = x_0!|\!)$$

via logical variables we can refer to initial values

Hoare Calculus

A Calculus for Program Verification

- aim: syntax directed calculus to reason about programs
- Hoare calculus separates reasoning on programs from logical reasoning (arithmetic, ...)
- present calculus as overview now, then explain single rules

$$\frac{\vdash (\!|\varphi|\!)\, C_1\, (\!|\eta|\!) \quad \vdash (\!|\eta|\!)\, C_2\, (\!|\psi|\!)}{\vdash (\!|\varphi|\!)\, C_1; C_2\, (\!|\psi|\!)} \text{ composition}$$

$$\frac{}{\vdash (\!|\varphi[x/e]|\!)\, x := e\, (\!|\varphi|\!)} \text{ assignment}$$

$$\frac{\vdash (\!|\varphi \wedge b|\!)\, C_1\, (\!|\psi|\!) \quad \vdash (\!|\varphi \wedge \neg b|\!)\, C_2\, (\!|\psi|\!)}{\vdash (\!|\varphi|\!)\, \mathtt{if}\ b\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2\, (\!|\psi|\!)} \text{ if-then-else}$$

$$\frac{\vdash (\!|\varphi \wedge b|\!)\, C\, (\!|\varphi|\!)}{\vdash (\!|\varphi|\!)\, \mathtt{while}\ b\ C\, (\!|\varphi \wedge \neg b|\!)} \text{ while}$$

$$\frac{\models \varphi \longrightarrow \varphi' \quad \vdash (\!|\varphi'|\!)\, C\, (\!|\psi'|\!) \quad \models \psi' \longrightarrow \psi}{\vdash (\!|\varphi|\!)\, C\, (\!|\psi|\!)} \text{ implication}$$

- read rules bottom up: in order to get lower part, prove upper part

Composition Rule

$$\frac{\vdash (\!|\varphi|\!)\, C_1\, (\!|\eta|\!) \quad \vdash (\!|\eta|\!)\, C_2\, (\!|\psi|\!)}{\vdash (\!|\varphi|\!)\, C_1; C_2\, (\!|\psi|\!)}\ \text{composition}$$

- applicability: whenever command is sequential composition $C_1; C_2$
- precondition is φ and aim is to show that ψ holds after execution
- rationale: find some midcondition η such that execution of C_1 guarantees η, which can then be used as precondition to conclude ψ after execution of C_2
- automation: finding suitable η is usually automatic, see later slides

Assignment Rule

$$\frac{}{\vdash (\!|\varphi[x/e]|\!)\; x := e\; (\!|\varphi|\!)}\ \text{assignment}$$

- applicability: whenever command is an assignment $x := e$
- to prove φ after execution, show $\varphi[x/e]$ before execution
- substitution seems to be on wrong side
  - effect of assignment is substitution x/e, so shouldn't rule be $\vdash (\!|\varphi|\!)\; x := e\; (\!|\varphi[x/e]|\!)$?
    No, this reversed rule would be wrong
    - assume before executing $x := 5$, the value of x is 6
    - before execution $\varphi = (x = 6)$ is satisfied, but after execution $\varphi[x/e] = (5 = 6)$ is not satisfied
- correct argumentation works as follows
  - if we want to ensure φ after the assignment then we need to ensure that the resulting situation $(\varphi[x/e])$ holds before
  - correct examples
    - $\vdash (\!|2 = 2|\!)\; x := 2\; (\!|x = 2|\!)$
    - $\vdash (\!|2 = 4|\!)\; x := 2\; (\!|x = 4|\!)$
    - $\vdash (\!|2 - y > 2^2|\!)\; x := 2\; (\!|x - y > x^2|\!)$
- applying rule is easy when read from right to left: just substitute

If-Then-Else Rule

$$\frac{\vdash (\!|\varphi \wedge b|\!)\, C_1\, (\!|\psi|\!) \qquad \vdash (\!|\varphi \wedge \neg b|\!)\, C_2\, (\!|\psi|\!)}{\vdash (\!|\varphi|\!)\ \texttt{if}\ b\ \texttt{then}\ C_1\ \texttt{else}\ C_2\ (\!|\psi|\!)}\ \text{if-then-else}$$

- applicability: whenever command is an if-then-else
- effect:
  - the preconditions in the two branches are strengthened by adding the corresponding (negated) condition b of the if-then-else
  - often the addition of b and $\neg b$ is crucial to be able to perform the proofs for the Hoare-triples of C_1 and C_2, respectively
- rationale: if b is true in some state, then the execution will choose C_1 and we can add b as additional assumption; similar for other case
- applying rule is trivial from right to left

While Rule

$$\frac{\vdash (\!|\varphi \wedge b|\!)\; C\; (\!|\varphi|\!)}{\vdash (\!|\varphi|\!)\; \texttt{while}\; b\; C\; (\!|\varphi \wedge \neg b|\!)}\;\text{while}$$

- applicability: only rule that handles while-loop
- key ingredient: loop invariant φ
- rationale
  - φ is precondition, so in particular satisfied before loop execution
  - $\vdash (\!|\varphi \wedge b|\!)\, C\, (\!|\varphi|\!)$ ensures, that when entering the loop, φ will be satisfied after one execution of the loop body C
  - in total, φ will be satisfied after each loop iteration
  - hence, when leaving the loop, φ and $\neg b$ are satisfied
  - while-rule does not enforce termination, partial correctness!
- automation
  - not automatic, since usually φ is not provided and postcondition is not of form $\varphi \wedge \neg b$; example: $\vdash (\!|x = x_0 \wedge x \geq 0|\!)\; \mathit{Fact}\; (\!|y = x_0!|\!)$
  - finding suitable φ is hard and needs user guidance

Implication Rule

$$\frac{\models \varphi \longrightarrow \varphi' \quad \vdash (\!|\varphi'|\!)\, C\, (\!|\psi'|\!) \quad \models \psi' \longrightarrow \psi}{\vdash (\!|\varphi|\!)\, C\, (\!|\psi|\!)}\ \text{implication}$$

- applicability: every command; does not change command
- rationale: weakening precondition or strengthening postcondition is sound
- remarks
  - only rule which does not decompose commands
  - application relies on prover for underlying logic, i.e., one which can prove implications
  - three main applications
    - simplify conditions that arise from applying other rules in order to get more readable proofs, e.g., replace $x + 1 = y - 2$ by $x = y - 3$
    - prepare invariants, e.g., change postcondition from ψ to some formula ψ' of form $\chi \wedge \neg b$
    - core reasoning engine when closing proofs for while-loops in proof tableaux, see later slides

Example Proof

$$
\dfrac{prf_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \llparenthesis (y \cdot x) \cdot (x-1)! = x_0! \land x - 1 \geq 0 \rrparenthesis\ \mathtt{y\ :=\ y\ *\ x}\ \llparenthesis y \cdot (x-1)! = x_0! \land x - 1 \geq 0 \rrparenthesis}}{\vdash \llparenthesis y \cdot x! = x_0! \land x \geq 0 \land x \neq 0 \rrparenthesis\ \mathtt{y\ :=\ y\ *\ x}\ \llparenthesis y \cdot (x-1)! = x_0! \land x - 1 \geq 0 \rrparenthesis} \qquad prf_2}{\vdash \llparenthesis y \cdot x! = x_0! \land x \geq 0 \land x \neq 0 \rrparenthesis\ \mathtt{y\ :=\ y\ *\ x;\ x\ :=\ x\ -\ 1}\ \llparenthesis y \cdot x! = x_0! \land x \geq 0 \rrparenthesis}}{\vdash \llparenthesis y \cdot x! = x_0! \land x \geq 0 \rrparenthesis\ \mathtt{while\ x\ !=\ 0\ \{y\ :=\ y\ *\ x;\ x\ :=\ x\ -\ 1\}}\ \llparenthesis y \cdot x! = x_0! \land x \geq 0 \land \neg x \neq 0 \rrparenthesis}}{\vdash \llparenthesis y \cdot x! = x_0! \land x \geq 0 \rrparenthesis\ \mathtt{while\ x\ !=\ 0\ \{y\ :=\ y\ *\ x;\ x\ :=\ x\ -\ 1\}}\ \llparenthesis y = x_0! \rrparenthesis}}{\vdash \llparenthesis x = x_0 \land x \geq 0 \rrparenthesis\ \mathtt{y\ :=\ 1;while\ x\ !=\ 0\ \{y\ :=\ y\ *\ x;\ x\ :=\ x\ -\ 1\}}\ \llparenthesis y = x_0! \rrparenthesis}
$$

where prf_1 is the following proof

$$
\dfrac{\dfrac{}{\vdash \llparenthesis 1 \cdot x! = x_0! \land x \geq 0 \rrparenthesis\ \mathtt{y\ :=\ 1}\ \llparenthesis y \cdot x! = x_0! \land x \geq 0 \rrparenthesis}}{\vdash \llparenthesis x = x_0 \land x \geq 0 \rrparenthesis\ \mathtt{y\ :=\ 1}\ \llparenthesis y \cdot x! = x_0! \land x \geq 0 \rrparenthesis}
$$

and prf_2 is the following proof

$$
\dfrac{}{\vdash \llparenthesis y \cdot (x-1)! = x_0! \land x - 1 \geq 0 \rrparenthesis\ \mathtt{x\ :=\ x\ -\ 1}\ \llparenthesis y \cdot x! = x_0! \land x \geq 0 \rrparenthesis}
$$

- only creative step: invention of loop invariant $y \cdot x! = x_0! \land x \geq 0$
- quite unreadable, introduce proof tableaux

Proof Tableaux

Problems in Presentation of Hoare Calculus

- proof trees become quite large even for small examples
- reason: lots of duplication, e.g., in composition rule

$$\frac{\vdash (\!|\varphi|\!)\, C_1\, (\!|\eta|\!) \quad \vdash (\!|\eta|\!)\, C_2\, (\!|\psi|\!)}{\vdash (\!|\varphi|\!)\, C_1; C_2\, (\!|\psi|\!)} \text{ composition}$$

  every formula φ, η, ψ occurs twice

- aim: develop better representation of Hoare-calculus proofs

Proof Tableaux

- main ideas
  - write program commands line-by-line
  - interleave program commands with midconditions
- structure

$$
\begin{array}{l}
\qquad (\!|\varphi_0|\!) \\
C_1; \\
\qquad (\!|\varphi_1|\!) \\
C_2; \\
\qquad (\!|\varphi_2|\!) \\
\dots \\
C_n \\
\qquad (\!|\varphi_n|\!)
\end{array}
$$

  where none of the C_i is a sequential execution
- idea: each midcondition φ_i should hold after execution of C_i

Weakest Preconditions

$$
\begin{array}{ll}
 & (\!|\varphi_i|\!) \\
C_{i+1}; & \\
 & (\!|\varphi_{i+1}|\!)
\end{array}
$$

- problem: how to find all the midconditions φ_i?
- solution
  - assume φ_{i+1} (and of course C_{i+1}) is given
  - then try to compute φ_i as weakest precondition,
    i.e., φ_i should be logically weakest formula satisfying

$$\models (\!|\varphi_i|\!)\ C_i\ (\!|\varphi_{i+1}|\!)$$

  - we will see, that such weakest preconditions can for many commands be computed automatically

Constructing the Proof Tableau

- aim: verify $\vdash (\!|\varphi_0'|\!)\, C_1; \ldots; C_n\, (\!|\varphi_n|\!)$
- approach: compute formulas $\varphi_{n-1}, \ldots, \varphi_0$, e.g., by taking weakest preconditions

$$
\begin{array}{rl}
 & (\!|\varphi_0|\!) \\
C_1; & \\
 & (\!|\varphi_1|\!) \\
\ldots & \\
C_{n-1}; & \\
 & (\!|\varphi_{n-1}|\!) \\
C_n & \\
 & (\!|\varphi_n|\!)
\end{array}
$$

  and check $\models \varphi_0' \longrightarrow \varphi_0$
  this last check corresponds to an application of the implication-rule
- next: consider the various commands how to compute a suitable formula φ_i given C_{i+1} and φ_{i+1}

Constructing the Proof Tableau – Assignment

- for the assignment, the weakest precondition is computed via

$$\begin{array}{l} \qquad (\!|\varphi[x/e]|\!) \\ x := e \\ \qquad (\!|\varphi|\!) \end{array}$$

- application is completely automatic: just substitute

Constructing the Proof Tableau – Implication

- represent implication-rule by writing two consecutive formulas

$$(\!|\psi|\!)$$
$$(\!|\varphi|\!)$$

  whenever $\models \psi \longrightarrow \varphi$

- application
  - simplify formulas
  - close proof tableau at the top, to turn given precondition into computed formula at top of program, e.g., $\models \varphi_0' \longrightarrow \varphi$ on slide 22
- example proof of $\vdash (\!|y = 2|\!)$ `y := y * y; x := y + 1` $(\!|x = 5|\!)$

$(\!|y = 2|\!)$

$(\!|y \cdot y = 4|\!)$ (closing proof tableau at top)

`y := y * y`

$(\!|y = 4|\!)$ (optional simplification step)

$(\!|y + 1 = 5|\!)$

`x := y + 1`

$(\!|x = 5|\!)$

Example with Destructive Updates

- assume we want to calculate $u = x + y$ via the following program P

$$\begin{array}{rl} & (\!|\mathsf{true}|\!) \\ & (\!|x + y = x + y|\!) \\ \texttt{z := x} & \\ & (\!|z + y = x + y|\!) \\ \texttt{z := z + y} & \\ & (\!|z = x + y|\!) \\ \texttt{u := z} & \\ & (\!|u = x + y|\!) \end{array}$$

- the midconditions have been inserted fully automatic
- hence we easily conclude $\vdash (\!|\mathsf{true}|\!)\ P\ (\!|u = x + y|\!)$
- note: although the tableau is constructed bottom-up, it also makes sense to read it top-down

An Invalid Example

- consider the following invalid tableau

$$\begin{array}{rl} & (\!|\text{true}|\!) \\ & (\!|x + 1 = x + 1|\!) \\ \texttt{x := x + 1} & \\ & (\!|x = x + 1|\!) \end{array}$$

- if the tableau were okay, then the result would be the arithmetic property $x = x + 1$, a formula that does not hold for any number x
- problem in tableau
  - assignment rule was not applied correctly
  - reason: substitution has to replace all variables
- corrected version

$$\begin{array}{rl} & (\!|x + 1 = (x + 1) + 1|\!) \\ \texttt{x := x + 1} & \\ & (\!|x = x + 1|\!) \end{array}$$

Constructing the Proof Tableau – If-Then-Else

- aim: calculate φ such that

$$\vdash (\!|\varphi|\!)\ \texttt{if}\ b\ \texttt{then}\ C_1\ \texttt{else}\ C_2\ (\!|\psi|\!)$$

  can be derived
- applying our procedure recursively, we get
  - formula φ_1 such that $\vdash (\!|\varphi_1|\!)\ C_1\ (\!|\psi|\!)$ is derivable
  - formula φ_2 such that $\vdash (\!|\varphi_2|\!)\ C_2\ (\!|\psi|\!)$ is derivable
- then weakest precondition for if-then-else is formula

$$\varphi := (b \longrightarrow \varphi_1) \wedge (\neg b \longrightarrow \varphi_2)$$

- formal justification that φ is sound

$$\dfrac{\dfrac{\vdash (\!|\varphi_1|\!)\ C_1\ (\!|\psi|\!)}{\vdash (\!|\varphi \wedge b|\!)\ C_1\ (\!|\psi|\!)} \qquad \dfrac{\vdash (\!|\varphi_2|\!)\ C_2\ (\!|\psi|\!)}{\vdash (\!|\varphi \wedge \neg b|\!)\ C_2\ (\!|\psi|\!)}}{\vdash (\!|\varphi|\!)\ \texttt{if}\ b\ \texttt{then}\ C_1\ \texttt{else}\ C_2\ (\!|\psi|\!)}$$

Example with If-Then-Else

- consider non-optimal code to compute the successor

$$(\!|\mathsf{true}|\!)$$

$$(\!|((x+1)-1=0 \longrightarrow 1=x+1) \wedge ((x+1)-1 \neq 0 \longrightarrow x+1=x+1)|\!)$$

```
a := x + 1;
```

$$(\!|(a-1=0 \longrightarrow 1=x+1) \wedge (a-1 \neq 0 \longrightarrow a=x+1)|\!)$$

```
if (a - 1 = 0) then {
```

$$(\!|1=x+1|\!)$$

```
  y := 1
```

$$(\!|y=x+1|\!)$$ (formula copied to end of then-branch)

```
} else {
```

$$(\!|a=x+1|\!)$$

```
  y := a
```

$$(\!|y=x+1|\!)$$ (formula copied to end of else-branch)

```
}
```

$$(\!|y=x+1|\!)$$

- insertion of midconditions is completely automatic
- large formula obtained in 2nd line must be proven in underlying logic

Applying the While Rule

$$\frac{\vdash (\!|\eta \wedge b|\!)\ C\ (\!|\eta|\!)}{\vdash (\!|\eta|\!)\ \texttt{while}\ b\ C\ (\!|\eta \wedge \neg b|\!)}\ \text{while}$$

- let us consider applicability in combination with implication-rule for arbitrary setting: how to derive the following?

  $$\vdash (\!|\varphi|\!)\ \texttt{while}\ b\ C\ (\!|\psi|\!)$$

  solution: find invariant η such that

  - $\models \varphi \longrightarrow \eta$ — precondition implies invariant
  - $\vdash (\!|\gamma|\!)\ C\ (\!|\eta|\!)$ — handle loop body recursively, produces γ
  - $\models \eta \wedge b \longrightarrow \gamma$ — η is indeed invariant
  - $\models \eta \wedge \neg b \longrightarrow \psi$ — invariant and $\neg b$ implies postcondition
- notes
  - invariant η has to be satisfied at beginning and end of loop-body, but not in between
  - invariant often captures the core of an algorithm:
    it describes connection between variables throughout execution
  - finding invariant is not automatic, but for seeing the connection
    it often helps to execute the loop a few rounds

Applying the While Rule – Soundness

$$\frac{\vdash (\!|\eta \wedge b|\!)\; C\; (\!|\eta|\!)}{\vdash (\!|\eta|\!)\; \texttt{while}\; b\; C\; (\!|\eta \wedge \neg b|\!)}\ \text{while}$$

- let us consider applicability in combination with implication-rule for arbitrary setting: how to derive the following?

  $$\vdash (\!|\varphi|\!)\; \texttt{while}\; b\; C\; (\!|\psi|\!)$$

  solution: find invariant η such that

  - $\models \varphi \longrightarrow \eta$ — precondition implies invariant
  - $\vdash (\!|\gamma|\!)\; C\; (\!|\eta|\!)$ — handle loop body recursively, produces γ
  - $\models \eta \wedge b \longrightarrow \gamma$ — η is indeed invariant
  - $\models \eta \wedge \neg b \longrightarrow \psi$ — invariant and $\neg b$ implies postcondition
- soundness proof

$$\cfrac{\cfrac{\cfrac{\vdash (\!|\gamma|\!)\; C\; (\!|\eta|\!)}{\vdash (\!|\eta \wedge b|\!)\; C\; (\!|\eta|\!)}}{\vdash (\!|\eta|\!)\; \texttt{while}\; b\; C\; (\!|\eta \wedge \neg b|\!)}}{\vdash (\!|\varphi|\!)\; \texttt{while}\; b\; C\; (\!|\psi|\!)}$$

Schema to Find Loop Invariant

- to create a Hoare-triple for a while-loop

$$\vdash (\!|\varphi|\!)\ \texttt{while}\ b\ C\ (\!|\psi|\!)$$

  find η such that
  - $\models \varphi \longrightarrow \eta$ — precondition implies invariant
  - $\vdash (\!|\gamma|\!)\ C\ (\!|\eta|\!)$ — handle loop body recursively, produces γ
  - $\models \eta \wedge b \longrightarrow \gamma$ — η is invariant
  - $\models \eta \wedge \neg b \longrightarrow \psi$ — invariant and $\neg b$ implies postcondition
- approach to find η
  1. guess initial η, e.g., based on a few loop executions
  2. check $\models \varphi \longrightarrow \eta$ and $\models \eta \wedge \neg b \longrightarrow \psi$; if not successful modify η
  3. compute γ by bottom-up generation of $\vdash (\!|\gamma|\!)\ C\ (\!|\eta|\!)$
  4. check $\models \eta \wedge b \longrightarrow \gamma$
  5. if last check is successful, proof is done
  6. otherwise, adjust η
- note: if φ is not known for checking $\models \varphi \longrightarrow \eta$, then instead perform bottom-up propagation of commands before while-loop (starting with η) and then use precondition of whole program

Verification of Factorial Program – Initial Invariant

- program P: `y := 1; while x > 0 {y := y * x; x := x - 1}`
- aim: $\vdash (\!|x = x_0 \wedge x \geq 0|\!)\, P\, (\!|y = x_0!|\!)$
- for guessing initial invariant, execute a few iterations to compute $6!$

iteration	x_0	x	y	$x!$
0	6	6	1	720
1	6	5	6	120
2	6	4	30	24
3	6	3	120	6
4	6	2	360	2
5	6	1	720	1

  observations
  - column $x!$ was added since computing $x!$ is aim
  - multiplication of y and $x!$ stays identical: $y \cdot x! = x_0!$
  - hence use $y \cdot x! = x_0!$ as initial candidate of invariant
- alternative reasoning with symbolic execution
  - in y we store $x_0 \cdot (x_0 - 1) \cdot \ldots \cdot (x + 1) = x_0!/x!$, so multiplying with $x!$ we get $y \cdot x! = x_0!$

Verification of Factorial Program – Testing Initial Invariant

- initial invariant: $\eta = (y \cdot x! = x_0!)$
- potential proof tableau

```
                (| x = x0 ∧ x ≥ 0 |)
                (| 1 · x! = x0! |)            (implication verified)
y := 1;
                (| η |)
while (x > 0) {
                (| η ∧ x > 0 |)

  y := y * x;

  x := x - 1
                (| η |)
}
                (| η ∧ ¬ x > 0 |)
                (| y = x0! |)                 (implication does not hold)
```

- problem: condition $\neg\, x > 0$ $(x \le 0)$ does not enforce $x = 0$ at end

Verification of Factorial Program – Strengthening Invariant

- strengthened invariant: $\eta = (y \cdot x! = x_0! \wedge x \geq 0)$
- potential proof tableau

$$(\!|x = x_0 \wedge x \geq 0|\!)$$

$$(\!|1 \cdot x! = x_0! \wedge x \geq 0|\!) \qquad \text{(implication verified)}$$

```
y := 1;
```

$$(\!|\eta|\!)$$

```
while (x > 0) {
```

$$(\!|\eta \wedge x > 0|\!)$$

$$(\!|(y \cdot x) \cdot (x-1)! = x_0! \wedge x - 1 \geq 0|\!) \qquad \text{(implication verified)}$$

```
  y := y * x;
```

$$(\!|y \cdot (x-1)! = x_0! \wedge x - 1 \geq 0|\!)$$

```
  x := x - 1
```

$$(\!|\eta|\!)$$

```
}
```

$$(\!|\eta \wedge \neg x > 0|\!)$$

$$(\!|y = x_0!|\!) \qquad \text{(implication verified)}$$

- proof completed, since all implications verified (e.g. by SMT solver)

Larger Example – Minimal-Sum Section

- assume extension of programming language: read-only arrays (writing into arrays requires significant extension of calculus)
- user is responsible for proper array access
- problem definition
  - given array $a[0], \dots, a[n-1]$ of length n, a section of a is a continuous block $a[i], \dots, a[j]$ with $0 \leq i \leq j < n$
  - define $S_{i,j}$ as sum of section
    $$S_{i,j} := a[i] + \cdots + a[j]$$
  - section (i, j) is minimal, if $S_{i,j} \leq S_{i',j'}$ for all sections (i', j') of a
- example: consider array $[-7, 15, -1, 3, 15, -6, 4, -5]$
  - $[3, 15, -6]$ and $[-6]$ are sections, but $[3, -6, 4]$ is not
  - there are two minimal-sum sections: $[-7]$ and $[-6, 4, -5]$

Minimal-Sum Section – Tasks

- write a program that computes sum of minimal section
- write a specification that makes "compute sum of minimal section" formal
- show that program satisfies the formal specification

Minimal-Sum Section – Challenges

- trivial algorithm
  - compute all sections ($O(n^2)$)
  - compute all sums of these sections and find the minimum
  - results in $O(n^3)$ algorithm
- aim: $O(n)$-algorithm which reads the array only once
- consequence: proof required that it is not necessary to explicitly compute all $O(n^2)$ sections
- example: consider array $[-8, 3, -65, 20, 45, -100, -8, 17, -4, -14]$
  - when reading from left-to-right a promising candidate might be $[-8, 3, -65]$, but there also is the later $[-100, -8]$, so how to decide what to take?

Minimal-Sum Section – Algorithm

- idea of algorithm
  - k: index that traverses array from left-to-right
  - s: minimal-sum of all sections seen so far
  - t: minimal-sum of all sections that end at position $k-1$
- algorithm *Min_Sum*

```
k := 1;
t := a[0];
s := a[0];
while (k != n) {
  t := min(t + a[k], a[k]);
  s := min(s, t);
  k := k + 1
}
```

- correctness not obvious, so let us better prove it

Minimal-Sum Section – Specification

- we split the specification in two parts via two Hoare-triples
  - Sp_1 specifies that the value of s is smaller than the sum of any section

$$(\!|\mathsf{true}|\!)\ \mathit{Min\_Sum}\ (\!|\forall i, j.\ 0 \leq i \leq j < n \longrightarrow s \leq S_{i,j}|\!)$$

  - Sp_2 specifies that there exists some section whose sum is s

$$(\!|\mathsf{true}|\!)\ \mathit{Min\_Sum}\ (\!|\exists i, j.\ 0 \leq i \leq j < n \wedge s = S_{i,j}|\!)$$

Minimal-Sum Section – Proving Sp_1

```
k := 1;
t := a[0];
s := a[0];
while (k != n) {
  t := min(t + a[k], a[k]);
  s := min(s, t);
  k := k + 1
}
```

$$Sp_1 : (\!|\mathsf{true}|\!)\ \mathit{Min\_Sum}\ (\!|\forall i, j.\ 0 \leq i \leq j < n \longrightarrow s \leq S_{i,j}|\!)$$

- find candidate invariant
  - invariant often similar to postcondition
  - invariant expresses relationships that are valid at beginning of each loop-iteration
- suitable invariant is $Inv_1(s, k)$ defined as

$$\forall i, j.\ 0 \leq i \leq j < k \longrightarrow s \leq S_{i,j}$$

$(\!|\, \mathit{Inv}_1(a[0], 1) \,|\!)$ (true statement)

```
k := 1;
```

$(\!|\, \mathit{Inv}_1(a[0], k) \,|\!)$

```
t := a[0];
```

$(\!|\, \mathit{Inv}_1(a[0], k) \,|\!)$

```
s := a[0];
```

$(\!|\, \mathit{Inv}_1(s, k) \,|\!)$

```
while (k != n) {
```

$(\!|\, \mathit{Inv}_1(s, k) \wedge k \neq n \,|\!)$

$(\!|\, \mathit{Inv}_1(\min(s, \min(t + a[k], a[k])), k + 1) \,|\!)$ (does not hold, no info on t)

```
  t := min(t + a[k], a[k]);
```

$(\!|\, \mathit{Inv}_1(\min(s, t), k + 1) \,|\!)$

```
  s := min(s, t);
```

$(\!|\, \mathit{Inv}_1(s, k + 1) \,|\!)$

```
  k := k + 1;
```

$(\!|\, \mathit{Inv}_1(s, k) \,|\!)$

```
}
```

$(\!|\, \mathit{Inv}_1(s, k) \wedge \neg k \neq n \,|\!)$

$(\!|\, \mathit{Inv}_1(s, n) \,|\!)$ (implication verified)

Minimal-Sum Section – Strengthening Invariant

```
k := 1;
t := a[0];
s := a[0];
while (k != n) {
  t := min(t + a[k], a[k]);
  s := min(s, t);
  k := k + 1
}
```

$$Sp_1 : (\!|\mathsf{true}|\!)\ \mathit{Min\_Sum}\ (\!|\forall i, j.\ 0 \leq i \leq j < n \longrightarrow s \leq S_{i,j}|\!)$$

- suitable invariant for s is $Inv_1(s, k)$ defined as

$$\forall i, j.\ 0 \leq i \leq j < k \longrightarrow s \leq S_{i,j}$$

- define similar invariant for t: $Inv_2(t, k)$ defined as

$$\forall i.\ 0 \leq i < k \longrightarrow t \leq S_{i,k-1}$$

- now try strengthened invariant $Inv_1(s, k) \wedge Inv_2(t, k)$

$(\!| \mathit{Inv}_1(a[0], 1) \wedge \mathit{Inv}_2(a[0], 1) |\!)$ (true statement)

```
k := 1;
```

$(\!| \mathit{Inv}_1(a[0], k) \wedge \mathit{Inv}_2(a[0], k) |\!)$

```
t := a[0];
```

$(\!| \mathit{Inv}_1(a[0], k) \wedge \mathit{Inv}_2(t, k) |\!)$

```
s := a[0];
```

$(\!| \mathit{Inv}_1(s, k) \wedge \mathit{Inv}_2(t, k) |\!)$

```
while (k != n) {
```

$(\!| \mathit{Inv}_1(s, k) \wedge \mathit{Inv}_2(t, k) \wedge k \neq n |\!)$

$(\!| \mathit{Inv}_1(\min(s, \min(t + a[k], a[k])), k + 1) \wedge \mathit{Inv}_2(\min(t + a[k], a[k]), k + 1) |\!)$ (implication verified)

```
  t := min(t + a[k], a[k]);
```

$(\!| \mathit{Inv}_1(\min(s, t), k + 1) \wedge \mathit{Inv}_2(t, k + 1) |\!)$

```
  s := min(s, t);
```

$(\!| \mathit{Inv}_1(s, k + 1) \wedge \mathit{Inv}_2(t, k + 1) |\!)$

```
  k := k + 1;
```

$(\!| \mathit{Inv}_1(s, k) \wedge \mathit{Inv}_2(t, k) |\!)$

```
}
```

$(\!| \mathit{Inv}_1(s, k) \wedge \mathit{Inv}_2(t, k) \wedge \neg k \neq n |\!)$

$(\!| \mathit{Inv}_1(s, n) |\!)$ (implication verified)

Minimal-Sum Section – Proving the Implications

- invariants
  - $Inv_1(s,k) := \forall i,j.\ 0 \leq i \leq j < k \longrightarrow s \leq S_{i,j}$
  - $Inv_2(t,k) := \forall i.\ 0 \leq i < k \longrightarrow t \leq S_{i,k-1}$
- implications
  - true $\longrightarrow$ *Inv*$_1(a[0],1) \wedge$ *Inv*$_2(a[0],1)$
    - because of the conditions of the quantifiers, by fixing $k = 1$ we only have to consider section $(0,0)$, i.e, we show $a[0] \leq S_{0,0} = a[0]$
  - let $0 < k < n$ where n is length of array a; then *Inv*$_1(s,k) \wedge$ *Inv*$_2(t,k) \wedge k \neq n$ implies both *Inv*$_2(\min(t + a[k], a[k]), k+1)$ and *Inv*$_1(\min(s, \min(t + a[k], a[k])), k+1)$;
    proof
    - pick any $0 \leq i < k + 1$; we show $\min(t + a[k], a[k])) \leq S_{i,k}$; if $i < k$ then $S_{i,k} = S_{i,k-1} + a[k]$, so we use *Inv*$_2(t,k)$ to get $t \leq S_{i,k-1}$ and thus $\min(t + a[k], a[k])) \leq t + a[k] \leq S_{i,k-1} + a[k] = S_{i,k}$;
      otherwise, $i = k$ and we have $\min(t + a[k], a[k]) \leq a[k] = S_{i,k}$
    - pick any $0 \leq i \leq j < k + 1$;
      we need to show $\min(s, \min(t + a[k], a[k])) \leq S_{i,j}$;
      if $j = k$ then the result follows from the previous statement;
      otherwise $j < k$ and the result follows from *Inv*$_1(s,k)$

Proof Tableaux – Summary

- we have proven soundness of non-trivial algorithm *Min_Sum*
- with gaps
  - we only proved Sp_1, but not Sp_2
  - lemma on previous slide demanded $0 < k < n$ which does not follow from loop-condition $k \neq n$; a proper fix would require a strengthened invariant which includes bounds on k
- main reasoning (proving the implications on previous slide) was done purely in logic with no reference to program
- such an approach is often conducted in verification of programs
  - there is a verification condition generator (VCG)
  - VCG converts assertions in programs (invariants) into logical formulas;
    here: Hoare-calculus handles program statements,
    verification conditions are instances of implication-rule
  - verification conditions are passed to SMT-solver, theorem prover, etc., to finally show correctness
  - problem: in case SMT-solver fails, user needs to understand failure to adapt invariants, assertions, etc.

Termination of Imperative Programs

Adding Termination to Calculus

- since while-loops are only source of non-termination in presented imperative language, it suffices to adjust the while-rule in the Hoare-calculus

all other Hoare-calculus rules can be used as before

- recall: total correctness = partial correctness + termination
- previous while-rule already proved partial correctness
- only task: extend existing while-rule to additionally prove termination
- idea of ensuring termination: use variants
  - a variant (or measure) is an integer expression;
  - this integer expression strictly decreases in every loop iteration and
  - at the same time the variant stays non-negative;
  - conclusion: there cannot be infinitely many loop iterations

A While-Rule For Total Correctness

- while-rule for partial correctness

$$\frac{\vdash (\!|\varphi \wedge b|\!)\; C\; (\!|\varphi|\!)}{\vdash (\!|\varphi|\!)\; \mathtt{while}\; b\; C\; (\!|\varphi \wedge \neg b|\!)}\;\text{while}$$

- extended while-rule for total correctness

$$\frac{\vdash (\!|\varphi \wedge b \wedge e_0 = e \geq 0|\!)\; C\; (\!|\varphi \wedge e_0 > e \geq 0|\!)}{\vdash (\!|\varphi \wedge e \geq 0|\!)\; \mathtt{while}\; b\; C\; (\!|\varphi \wedge \neg b|\!)}\;\text{while-total}$$

  where
  - e is variant expression with values before execution of C
  - e is (the same) variant expression with values after execution of C
  - e_0 is fresh logical variable, used to store the value of e before: $e_0 = e$
  - hence, postcondition $e_0 > e$ enforces decrease of e when executing C
  - non-negativeness is added three times, even in precondition of while
  - e is of type integer so that $SN\ \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x > y \geq 0\}$ can be used as underlying terminating relation: each loop iteration corresponds to a step $([\![e]\!]_{\alpha_{\text{before}}}, [\![e]\!]_{\alpha_{\text{after}}})$ in this relation

Applying While-Total

$$\frac{\vdash (\!|\varphi \wedge b \wedge e_0 = e \geq 0|\!)\, C\, (\!|\varphi \wedge e_0 > e \geq 0|\!)}{\vdash (\!|\varphi \wedge e \geq 0|\!)\, \texttt{while}\ b\ C\, (\!|\varphi \wedge \neg b|\!)} \text{ while-total}$$

- application
  - e_0 is fresh logical variable, so nothing to choose
  - variant e has to be chosen, but this is often easy
    - `while (x < 5) { ...  x := x + 1 ...}` is same as `while (5 - x > 0) { ...  x := x + 1 ...}`, so $e = 5 - x$
    - `while (y >= x) { ...  y := y - 2 ...}` is same as `while (y - x >= 0) { ...  y := y - 2 ...}`, so $e = y - x\ (+2)$
    - `while (x != y) { ...  y := y + 1 ...}` is same as `while (x - y != 0) { ...  y := y + 1 ...}`, so $e = x - y$
  - checking the condition is then easily possible via proof tableau, in the same way as for the while-rule for partial correctness
  - all side-conditions $e \geq 0$ can completely be eliminated by choosing $e = \max(0, e')$ for some e', but then proving $e_0 > e$ will become harder as it has to deal with $\max$
  - invariant φ can be taken unchanged from partial correctness proof

Total Correctness of Factorial Program

- red parts have been added for termination proof with variant $x - z$

$(\!|\mathsf{true} \wedge x \geq 0|\!)$ (new termination condition on x)

$(\!|1 = 0! \wedge x - 0 \geq 0|\!)$

```
y := 1;
```

$(\!|y = 0! \wedge x - 0 \geq 0|\!)$

```
z := 0;
```

$(\!|y = z! \wedge x - z \geq 0|\!)$ (new condition added)

```
while (x != z) {
```

$(\!|y = z! \wedge x \neq z \wedge e_0 = x - z \geq 0|\!)$ (new condition added)

$(\!|y \cdot (z+1) = (z+1)! \wedge e_0 > x - (z+1) \geq 0|\!)$ (more reasoning)

```
  z := z + 1;
```

$(\!|y \cdot z = z! \wedge e_0 > x - z \geq 0|\!)$

```
  y := y * z;
```

$(\!|y = z! \wedge e_0 > x - z \geq 0|\!)$ (new condition added)

```
}
```

$(\!|y = z! \wedge \neg x \neq z|\!)$

$(\!|y = x!|\!)$

Remarks on Total Correctness of Factorial Program

- precondition $x \geq 0$ was added automatically from termination proof
- in fact, the program does not terminate on negative inputs
- for factorial program (and other imperative programs) Hoare-calculus permits to prove local termination, i.e., termination on certain inputs
- in contrast, for functional program we always considered universal termination, i.e., termination of all inputs
- termination proofs can also be performed stand-alone (without partial correctness proof): just prove postcondition "true" with while-total-rule:

$$\vdash (\!|\varphi|\!)\; P\; (\!|\mathsf{true}|\!)$$

implies termination of P on inputs that satisfy φ, so

$$\vdash (\!|\mathsf{true}|\!)\; P\; (\!|\mathsf{true}|\!)$$

shows universal termination of P

Soundness of Hoare-Calculus

Soundness of Hoare-Calculus

- so far, we have two notions of soundness
  - $\models (\!|\varphi|\!)\, P\, (\!|\psi|\!)$: via semantic of imperative programs, i.e., whenever $\alpha \models \varphi$ and $(P, \alpha) \hookrightarrow^* (\texttt{skip}, \beta)$ then $\beta \models \psi$ must hold
  - $\vdash (\!|\varphi|\!)\, P\, (\!|\psi|\!)$: syntactic, what can be derived via Hoare-calculus rules
- missing: soundness of calculus, i.e.,

$$\vdash (\!|\varphi|\!)\, P\, (\!|\psi|\!) \text{ implies } \models (\!|\varphi|\!)\, P\, (\!|\psi|\!)$$

- formal proof is based on big-step semantics $\to$ (see exercises): $(P, \alpha) \hookrightarrow^* (\texttt{skip}, \beta)$ is turned into $(P, \alpha) \to \beta$
- soundness of the calculus is then established by the following property, which is proven by induction w.r.t. the Hoare-calculus rules for arbitrary α, β:

$$\vdash (\!|\varphi|\!)\, C\, (\!|\psi|\!) \longrightarrow \alpha \models \varphi \longrightarrow (C, \alpha) \to \beta \longrightarrow \beta \models \psi$$

Proving $\vdash (\!|\varphi|\!)\, C\, (\!|\psi|\!) \longrightarrow \alpha \models \varphi \longrightarrow (C, \alpha) \to \beta \longrightarrow \beta \models \psi$

Case 1: implication-rule

$\vdash (\!|\varphi|\!)\, C\, (\!|\psi|\!)$ since $\models \varphi \longrightarrow \varphi'$, $\vdash (\!|\varphi'|\!)\, C\, (\!|\psi'|\!)$, and $\models \psi' \longrightarrow \psi$

- IH: $\forall \alpha, \beta.\, \alpha \models \varphi' \longrightarrow (C, \alpha) \to \beta \longrightarrow \beta \models \psi'$
- assume $\alpha \models \varphi$ and $(C, \alpha) \to \beta$
- then by $\models \varphi \longrightarrow \varphi'$ conclude $\alpha \models \varphi'$
- in combination with IH get $\beta \models \psi'$
- with $\models \psi' \longrightarrow \psi$ conclude $\beta \models \psi$

Proving $\vdash (\!|\varphi|\!)\, C\, (\!|\psi|\!) \longrightarrow \alpha \models \varphi \longrightarrow (C, \alpha) \to \beta \longrightarrow \beta \models \psi$

Case 2: composition-rule

$\vdash (\!|\varphi|\!)\, C_1; C_2\, (\!|\psi|\!)$ since $\vdash (\!|\varphi|\!)\, C_1\, (\!|\eta|\!)$ and $\vdash (\!|\eta|\!)\, C_2\, (\!|\psi|\!)$

- IH-1: $\forall \alpha, \beta.\, \alpha \models \varphi \longrightarrow (C_1, \alpha) \to \beta \longrightarrow \beta \models \eta$
- IH-2: $\forall \alpha, \beta.\, \alpha \models \eta \longrightarrow (C_2, \alpha) \to \beta \longrightarrow \beta \models \psi$
- assume $\alpha \models \varphi$ and $(C_1; C_2, \alpha) \to \beta$
- from the latter and the definition of $\to$, there must be γ such that $(C_1, \alpha) \to \gamma$ and $(C_2, \gamma) \to \beta$
- by using IH-1 (choose α and γ in $\forall$), obtain $\gamma \models \eta$
- by using IH-2 (choose γ and β in $\forall$), obtain $\beta \models \psi$

Proving $\vdash (\!|\varphi|\!)\, C\, (\!|\psi|\!) \longrightarrow \alpha \models \varphi \longrightarrow (C, \alpha) \to \beta \longrightarrow \beta \models \psi$

Case 3: if-then-else-rule

$\vdash (\!|\varphi|\!)\, \texttt{if}\ b\ \texttt{then}\ C_1\ \texttt{else}\ C_2\, (\!|\psi|\!)$

since $\vdash (\!|\varphi \wedge b|\!)\, C_1\, (\!|\psi|\!)$ and $\vdash (\!|\varphi \wedge \neg b|\!)\, C_2\, (\!|\psi|\!)$

- IH-1: $\forall \alpha, \beta.\ \alpha \models \varphi \wedge b \longrightarrow (C_1, \alpha) \to \beta \longrightarrow \beta \models \psi$
- IH-2: $\forall \alpha, \beta.\ \alpha \models \varphi \wedge \neg b \longrightarrow (C_2, \alpha) \to \beta \longrightarrow \beta \models \psi$
- assume $\alpha \models \varphi$ and $(\texttt{if}\ b\ \texttt{then}\ C_1\ \texttt{else}\ C_2, \alpha) \to \beta$
- perform case analysis on $[\![b]\!]_\alpha$
- w.l.o.g. we only consider the case $[\![b]\!]_\alpha =$ true where
    - from $\alpha \models \varphi$ conclude $\alpha \models \varphi \wedge b$
    - from $(\texttt{if}\ b\ \texttt{then}\ C_1\ \texttt{else}\ C_2, \alpha) \to \beta$ conclude $(C_1, \alpha) \to \beta$
    - by using IH-1 get $\beta \models \psi$

Proving $\vdash (\!|\varphi|\!)\, C\, (\!|\psi|\!) \longrightarrow \alpha \models \varphi \longrightarrow (C, \alpha) \to \beta \longrightarrow \beta \models \psi$

Case 4: assignment-rule

$\vdash (\!|\varphi|\!)\, x := e\, (\!|\psi|\!)$ since $\varphi = \psi[x/e]$

- assume $\alpha \models \varphi$ and $(x := e, \alpha) \to \beta$
- by definition of $\to$, conclude $\beta = \alpha[x := [\![e]\!]_\alpha]$
- hence assumption $\alpha \models \varphi$ is equivalent to
  - $\alpha \models \psi[x/e]$ — by unrolling φ-equality
  - $\alpha[x := [\![e]\!]_\alpha] \models \psi$ — by substitution lemma for formulas
  - $\beta \models \psi$ — by unrolling β-equality

Proving $\vdash (\!|\varphi|\!)\, C\, (\!|\psi|\!) \longrightarrow \alpha \models \varphi \longrightarrow (C, \alpha) \to \beta \longrightarrow \beta \models \psi$

Case 5: while-rule

$\vdash (\!|\varphi|\!)\, \texttt{while}\ b\ C'\, (\!|\psi|\!)$ since $\vdash (\!|\varphi \wedge b|\!)\, C'\, (\!|\varphi|\!)$ and $\psi = \varphi \wedge \neg b$

- (outer) IH: $\forall \alpha, \beta.\, \alpha \models \varphi \wedge b \longrightarrow (C', \alpha) \to \beta \longrightarrow \beta \models \varphi$
- we now prove $\alpha \models \varphi \longrightarrow (\texttt{while}\ b\ C', \alpha) \to \beta \longrightarrow \beta \models \psi$
  by an inner induction on α w.r.t. $\to$, but for fixed b, C', β, φ, ψ
  - case 1: $(\texttt{while}\ b\ C', \alpha) \to \beta$
    since $[\![b]\!]_\alpha =$ false and $\beta = \alpha$
    - in this case conclude $\beta = \alpha \models \varphi \wedge \neg b = \psi$
  - case 2: $(\texttt{while}\ b\ C', \alpha) \to \beta$
    since $[\![b]\!]_\alpha =$ true, $(C', \alpha) \to \gamma$ and $(\texttt{while}\ b\ C', \gamma) \to \beta$
    - inner IH: $\gamma \models \varphi \longrightarrow \beta \models \psi$
    - assume $\alpha \models \varphi$
    - hence $\alpha \models \varphi \wedge b$
    - by outer IH (choose α and γ in $\forall$) get $\gamma \models \varphi$
    - then inner IH yields $\beta \models \psi$

Summary of Soundness of Hoare-Calculus

- since Hoare-calculus rules and semantics are formally defined, it is possible to verify soundness of the calculus
- proof requires inner induction for while-loop,
  since big-step semantics of while-command refers to itself
- here: only soundness of Hoare-calculus for partial correctness
- possible extension: total correctness
  - define semantic notion $\models_{total} (\!|\varphi|\!)\, C\, (\!|\psi|\!)$ stating total correctness
  - prove that Hoare-calculus with while-total is sound w.r.t. $\models_{total}$

Programming by Contract

Programming by Contract – Idea

- Hoare-triple $(\!|\varphi|\!)\, P\, (\!|\psi|\!)$ may be seen as a contract between supplier and consumer of program P
  - supplier insists that consumer invokes P only on states satisfying φ
  - supplier promises that after execution of P formula ψ holds
- validation of Hoare-triples with Hoare-calculus can be seen as validation of contracts for method- or procedure-calls

Example

- consider method where . . . is program *Fact* on slide 9

  `int factorial (int x) { int y; ...; return y }`

- example contract

```
method name:      factorial
input:            int x
output:           int
assumes:          x >= 0
guarantees:       result = x!
modifies only:    local variables
```

- remarks
  - return-value of method is referred to as `result` in contract
  - since x is local parameter (call-by-value) and y is local variable, there will be no impact on global variables;
  - for procedures and call-by-reference variables, one usually wants to know whether modifications take place

Modified Example

- consider procedure where `...` is program *Fact* on slide 9

  ```
  void factorial_proc (int x) { ... }
  ```

- example contract

  ```
  procedure name:   factorial_proc
  input:            int x
  assumes:          x >= 0
  guarantees:       y = x!
  modifies only:    y
  ```

- remarks
  - y is no longer local variable, but global
  - procedure has no return value
  - guarantees are expressed via global variables and parameters (and if required, logical variables)
  - modification of global variable y visible in contract

Invoking Methods

- assume we want to write method for binomial coefficients

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

  to compute chance of lotto-jackpot $1 : \binom{49}{6}$

- ```
  int binom (int n, int k) {
    return factorial(n) / (factorial(k) * factorial (n-k))
  }
  ```

- programming-by-contract also demands contracts for new methods
- in example, we need to ensure that preconditions of `factorial`-invocations are met

```
method name:      binom
inputs:           int n, int k
output:           int
assumes:          n >= 0, k >= 0, n >= k
guarantees:       result = n choose k
modifies only:    local variables
```
```

Programming by Contract – Advantages

- in the same way as methods help to structure larger programs, contracts for these methods help to verify larger programs
- reason: for verifying code invoking method m, it suffices to look at contract of m – without looking at implementation of m
- positive effects
  - add layer of abstraction
  - easy to change implementation of m as long as contract stays identical
  - verification becomes more modular
- example: for invocation of `min` in minimal-sum section it does not matter whether
  - `min` is built-in operator which is substituted as such, or
  - `min` is user-defined method that according to the contract computes the mathematical $\min$-operation
    implementation can be ignored for caller, but developer needs to verify it against contract

```
int min(int x, int y) {
  int z;
  if x <= y then z := x else z := y;
  return z }
```

Summary – Verification of Imperative Programs

- covered
  - syntax and semantic of small imperative programming language
  - Hoare-calculus to verify Hoare-triples $(\!|\varphi|\!)\ P\ (\!|\psi|\!)$
  - proof tableaux and automation:
    Hoare-calculus is VCG that converts program logic into implications (verification conditions) that must be shown in underlying logic
  - proofs are mostly automatic, except for loop invariants
  - soundness of Hoare-calculus
  - programming by contracts: abstract from concrete method-implementations, use contracts
- not covered
  - heap-access, references, arrays, etc.: extension to separation logic, memory model
  - bounded integers: reasoning engine for bit-vector-arithmetic
  - multi-threading