Homework

1. Determine satisfiability of the following LRA formula ϕ using DPLL(T) and the incremental simplex implementation from the lecture.[1] You should use standard Boolean unit propagation as well as theory propagation, i.e., determine sets of implied literals from a partial Boolean assignment via the theory solver. (2 P)

   Hint: For this example you never have to perform any guesses, i.e., do not apply the inference rule for a Boolean decision.

$$\begin{aligned}
\phi &:= c_1 \wedge c_2 \wedge c_3 \wedge (c_4 \vee c_5) \wedge (c_6 \vee c_7) \\
c_1 &:= 2x - 40y + 4z \leq 10 \\
c_2 &:= 15y + 4z \leq 15 \\
c_3 &:= x + y + z \leq 100 \\
c_4 &:= 3x + 14y + 7z \geq 450 \\
c_5 &:= -2x + 2y - z \geq 15 \\
c_6 &:= 2x - y + z > 20 \\
c_7 &:= 2x - 3y + 2z \geq 100
\end{aligned}$$

2. Farkas' lemma states that a set of non-strict inequalities is unsatisfiable if and only if there exists Farkas' coefficients (see slide 14).

   (a) Consider the following inequalities

   $$\underbrace{-x + 2y}_{s_1} \leq -2 \qquad \underbrace{3x + 2y}_{s_2} \leq 12 \qquad \underbrace{-2x - 3y}_{s_3} \leq -12$$

   where the variables range over $\mathbb{Q}$. Simplex finds these to be unsatisfiable where the final tableau is

   $$y = -\frac{2}{5}s_2 - \frac{3}{5}s_3 \qquad x = \frac{3}{5}s_2 + \frac{2}{5}s_3 \qquad s_1 = -\frac{7}{5}s_2 - \frac{8}{5}s_3$$

   with an assignment $v(x) = v(y) = v(s_1) = \frac{12}{5}$, $v(s_2) = 12$, $v(s_3) = -12$. Find Farkas' coefficients that prove unsatisfiability for these inequalities. (2 P)

   (b) Prove completeness of Faraks' lemma, by constructing an algorithm to extract Farkas' coefficients from any unsatisfiable simplex state. (3 P)

3. Using Farkas' lemma from the lecture, prove that the following variant of the lemma also holds. (3 P)

   Let $\varphi = (\ell_1 \leqslant b_1) \wedge (\ell_2 \leqslant b_2) \wedge \cdots \wedge (\ell_n \leqslant b_n)$ be a system of linear inequalities and $\ell < b$ a linear inequality, where $b_1, \ldots, b_n, b \in \mathbb{Q}$ and all $\ell_1, \ldots, \ell_n, \ell$ are linear-homogeneous expressions (i.e. they have the shape $\ell_i = \sum a_i x_i$). If φ is satisfiable, and $\varphi \to (\ell < b)$ is valid then there exist coefficients $c_1, \ldots, c_n \geqslant 0$ such that

   $$\sum_{i=1}^{n} c_i \ell_i = \ell \quad \text{and} \quad \sum_{i=1}^{n} c_i b_i \leqslant b$$

[1] http://cl-informatik.uibk.ac.at/teaching/ss24/cs/simplex.tgz