Homework

1. On slide 12 we showed that there is a LIA formula φ where any satisfying assignment contains a value of at least c^n. Design a formula which has a larger lower bound. (1 P)

2. Show that determining satisfiablility of a conjunction of linear integer constraints is NP-hard. (2 P)
   Hint: Use a reduction from CNF-SAT.

3. For some optimization problems it is required to consider *mixed* linear arithmetic problems, where the set of variables $\mathcal{V}$ is partitioned into $\mathcal{V}_\mathbb{Z} \uplus \mathcal{V}_\mathbb{Q}$, i.e., some variables are restricted to represent integers whereas the others are for rational numbers. For instance $\mathcal{V}_\mathbb{Z}$ might describe quantities which cannot be split, such as wheels, screws, etc., whereas $\mathcal{V}_\mathbb{Q}$ is used for amounts of fluids, for costs, etc.

   A mixed solution of a quantifier-free formula ϕ is an assignment $v : \mathcal{V} \to \mathbb{Q}$ that satisfies ϕ and where additionally $v(x) \in \mathbb{Z}$ for all $x \in \mathcal{V}_\mathbb{Z}$.

   (a) Modify the branch-and-bound algorithm for LIA so that it can treat mixed linear arithmetic constraints, i.e., write down pseudo-code. (2 P)

   (b) Consider the small model property of LIA.

   The existing proof translates a set of constraints with n variables given as a polyhedron $\{\vec{x} \mid A\vec{x} \leq \vec{b}\}$ with $A \in \mathbb{Z}^{m \times n}$ and $\vec{b} \in \mathbb{Z}^m$ into $\mathit{hull}(H) + \mathit{cone}(C)$ for some $H \subseteq \mathbb{Q}^n$ and $C \subseteq \mathbb{Z}^n$ where additionally some upper bound u is constructed in a way that all coefficients c of all vectors in $H \cup C$ are bounded: $|c| \leq u$.

   Afterwards it is shown how some arbitrary integral solution $\vec{x} \in \mathit{hull}(H) + \mathit{cone}(C)$ can be turned into a small integral solution $\vec{y} \in \mathit{hull}(H) + \mathit{cone}(C)$ where $|y_i|$ is bounded by some expression involving u and n for all coefficients y_i of $\vec{y}$.

   i. Provide an upper bound for $|y_i|$ depending on u and n. (1 P)

   ii. Does the transformation of the arbitrary integer solution $\vec{x}$ into the small integer solution $\vec{y}$ also work correctly in the mixed case with $\mathcal{V}_\mathbb{Z} = \{x_1, \dots, x_k\}$ and $\mathcal{V}_\mathbb{Q} = \{x_{k+1}, \dots, x_n\}$, i.e., when replacing the conditions of $\vec{x}, \vec{y}$ being integral by $\vec{x}, \vec{y} \in \mathbb{Z}^k \times \mathbb{Q}^{n-k}$? Explain your answer. (2 P)

4. Consider the construction on slide 21. Let $P = \{\vec{u} \mid A\vec{u} \leqslant \vec{b}\}$, and X and V be the sets of vectors as defined on slide 21. Prove $P \subseteq \mathit{cone}(X) + \mathit{hull}(V)$. You may assume the Farkas–Minkowsky–Weyl theorem and don't have to prove it. (2 P)