universität innsbruck

Constraint Solving	SS 2024	LVA 703305
Week 11		June 3, 2024

Homework

1. Use Nelson-Oppen to determine satisfiability of the following formula over EUF and LIA. For LIA, no decision procedure is necessary, just specify consequences of formulas on paper. (1 P)

$$1 \leq x \wedge x \leq 2 \wedge \mathsf{f}(1) = \mathsf{a} \wedge \mathsf{f}(x) = \mathsf{b} \wedge \mathsf{a} = \mathsf{b} + 2 \wedge \mathsf{f}(2) = \mathsf{f}(1) + 3$$

2. Prove that the deterministic Nelson-Oppen method is sound assuming soundness of the non-deterministic Nelson-Oppen method. More specifically, let T_1 and T_2 be convex theories with signature Σ_1 and Σ_2 respectively where $\Sigma_1 \cap \Sigma_2 = \{=\}$, and $\varphi = \varphi_1 \wedge \varphi_2$ a formula where φ_1 (φ_2) is a Σ_1 (Σ_2) formula. Prove that if the deterministic Nelson-Oppen returns satisfiable, then φ is satisfiable. (2 P)

3. Model the existence of a winning strategy for Gomoku[1] in QBF. For this you should encode the board as a set of propositional variables, and construct the following formulas: (2 P)
   (a) Define a formula $\varphi^i_{move}(b_1, b_2)$ which is true iff b_2 is a valid board after a move of player i on the board b_1, and a formula $\varphi_{win}(b)$ which is true iff some player has won on the board b.
   (b) Encode the following statements as QBF formulas
      - Player one can always win, independent of the moves played by player two.
      - Player two can always win, independent of the moves played by player one.

4. In this exercise we will prove that LRA is convex as defined on slide 11. Let $A\vec{x} \leqslant \vec{c}$ be an LRA problem where $\vec{x} = (x_1 \; x_2 \; \cdots x_n)^T$, and $E = \{x_{i_1} = x_{j_1}, \ldots, x_{i_m} = x_{j_m}\}$ is a set of equations over $x_1, \ldots, x_n$ such that $A\vec{x} \leqslant \vec{c} \implies \bigvee_{x_i = x_j \in E} x_i = x_j$.
   (a) First show that the set of solutions of an LRA problem is convex in the geometric sence. In other words, prove that for all $\vec{a}, \vec{b} \in \mathbb{R}^n$ we have (1 P)

   $$A\vec{a} \leqslant \vec{c} \wedge A\vec{b} \leqslant \vec{c} \quad \implies \quad \{\, (1-\lambda) \cdot \vec{a} + \lambda \cdot \vec{b} \mid 0 \leqslant \lambda \leqslant 1 \,\} \subseteq \{\vec{x} \mid A\vec{x} \leqslant \vec{c}\}$$

   (b) Use the statement from (a) to show that for any two points $\vec{a}, \vec{b} \in \{\vec{x} \mid A\vec{x} \leqslant \vec{c}\}$ there exists an equation $x_i = x_j \in E$ such that both $a_i = a_j$ and $b_i = b_j$ holds. (2 P)
   (c) Finally use the statement from (b) to show that $A\vec{x} \leqslant \vec{c} \implies x_i = x_j$ for some equation $x_i = x_j \in E$. (2 P)

[1] https://en.wikipedia.org/w/index.php?title=Gomoku&oldid=1223707331#Rules