universität innsbruck

SS 2024 lecture 2

Constraint Solving

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based on a previous course by Aart Middeldorp

Outline

1. **Summary of Previous Lecture**
2. **Conflict Graphs**
3. **NP-Completeness of SAT**
4. **SAT Reductions**
5. **Further Reading**

Theorem

propositional formula φ is valid $\iff$ $\neg\varphi$ is unsatisfiable

Definitions

- literal is atom p or negation $\neg p$ of atom
- clause is disjunction of literals
- conjunctive normal form (CNF) is conjunction of clauses
- disjunctive normal form (DNF) is disjunction of conjunctions of literals

Theorem

$\forall$ formula φ $\exists$ CNF ψ $\exists$ DNF χ such that $\varphi \equiv \psi \equiv \chi$

Remark

Tseitin's transformation is linear-time translation to equisatisfiable CNF

Definition (Abstract DPLL)

- states $M \parallel F$ consist of list M of (possibly annotated) non-complementary literals and CNF F
- transition rules
  - unit propagate $\qquad M \parallel F, C \vee l \quad \Longrightarrow \quad M\,l \parallel F, C \vee l$

    if $M \vDash \neg C$ and l is undefined in M
  - pure literal $\qquad M \parallel F \quad \Longrightarrow \quad M\,l \parallel F$

    if l occurs in F and l^c does not occur in F and l is undefined in M
  - decide $\qquad M \parallel F \quad \Longrightarrow \quad M\,l^d \parallel F$

    if l or l^c occurs in F and l is undefined in M

Definition (Abstract DPLL, cont'd)

- fail $\quad M \parallel F, C \implies \text{fail-state}$

  if $M \vDash \neg C$ and M contains no decision literals

- backtrack $\quad M \overset{d}{l} N \parallel F, C \implies M\, l^c \parallel F, C$

  if $M \overset{d}{l} N \vDash \neg C$ and N contains no decision literals

- backjump $\quad M \overset{d}{l} N \parallel F, C \implies M\, l' \parallel F, C$

  if $M \overset{d}{l} N \vDash \neg C$ and $\exists$ clause $C' \vee l'$ such that

  backjump clause

  - $F, C \vDash C' \vee l'$
  - $M \vDash \neg C'$
  - l' is undefined in M
  - l' or l'^c occurs in F or in $M \overset{d}{l} N$

Definition

basic DPLL $\mathcal{B}$ consists of transition rules unit propagate, decide, fail, backjump

Theorem

- there are no infinite derivations $\parallel F \Longrightarrow_{\mathcal{B}} S_1 \Longrightarrow_{\mathcal{B}} S_2 \Longrightarrow_{\mathcal{B}} \cdots$
- if $\parallel F \Longrightarrow_{\mathcal{B}} S_1 \Longrightarrow_{\mathcal{B}} \cdots \Longrightarrow_{\mathcal{B}} S_n \not\Longrightarrow_{\mathcal{B}}$ then
  1. $S_n =$ fail-state if and only if F is unsatisfiable
  2. $S_n = M \parallel F'$ only if F is satisfiable and $M \vDash F$

Outline

Problem: How to obtain backjump clauses

- backjump $M \overset{d}{l} N \parallel F, C \implies M l' \parallel F, C$

  if $M \overset{d}{l} N \vDash \neg C$ and ... (some more conditions; involves finding a backjump clause)
- situation: complicated looking rule; unclear how to obtain backjump clause

Problem: How to obtain backjump clauses

- backjump $$M\, \overset{d}{l}\, N \parallel F, C \quad \Longrightarrow \quad M\, l' \parallel F, C$$
  if $M\, \overset{d}{l}\, N \vDash \neg C$ and ... (some more conditions; involves finding a backjump clause)
- situation: complicated looking rule; unclear how to obtain backjump clause
- solution

Problem: How to obtain backjump clauses

- backjump $\quad M \overset{d}{l} N \parallel F, C \implies M l' \parallel F, C$

  if $M \overset{d}{l} N \models \neg C$ and ...(some more conditions; involves finding a backjump clause)
- situation: complicated looking rule; unclear how to obtain backjump clause
- solution
  - store information of applied rules (unit propagate, decide, ...) in conflict graph

Problem: How to obtain backjump clauses

- backjump $\quad M \overset{d}{l} N \parallel F, C \implies M l' \parallel F, C$

  if $M \overset{d}{l} N \vDash \neg C$ and ...(some more conditions; involves finding a backjump clause)
- situation: complicated looking rule; unclear how to obtain backjump clause
- solution
  - store information of applied rules (unit propagate, decide, ...) in conflict graph
  - cuts in conflict graphs separate conflict node from current decision literal and literals at earlier decision levels

Problem: How to obtain backjump clauses

- backjump $\quad M \overset{d}{l} N \parallel F, C \implies M\, l' \parallel F, C$

  if $M \overset{d}{l} N \vDash \neg C$ and . . . (some more conditions; involves finding a backjump clause)
- situation: complicated looking rule; unclear how to obtain backjump clause
- solution
  - store information of applied rules (unit propagate, decide, . . .) in conflict graph
  - cuts in conflict graphs separate conflict node from current decision literal and literals at earlier decision levels
  - cuts that correspond to unique implication points (UIPs) generate backjump clauses

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
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ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
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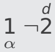
1
α

Example



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Example

$$\underset{\alpha}{1} \quad \overset{d}{\neg 2} \quad \underset{\beta}{\neg 3}$$

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$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}$$

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Example

$$1_{\alpha}\ \neg\overset{d}{2}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ \overset{d}{5}$$

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β	$2 \vee \neg 3$
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$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}$$

Example

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$$\underset{\alpha}{1}\ \neg\overset{d}{2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}$$

Example

α	1	$1_{\alpha}\ \neg\overset{d}{2}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ \overset{d}{5}\ \neg 6_{\zeta}\ 7_{\eta}\ \overset{d}{8}$ decision level 3
β	$2 \vee \neg 3$	
γ	$2 \vee \neg 4$	
δ	$3 \vee \neg 12 \vee \neg 13$	
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$	
ζ	$\neg 5 \vee \neg 6$	
η	$6 \vee 7$	
θ	$6 \vee \neg 12 \vee 14$	
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$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}$$

Example

$$1 \quad \overset{d}{\neg 2} \quad \neg 3 \quad \neg 4 \quad \overset{d}{5} \quad \neg 6 \quad 7 \quad \overset{d}{8} \quad 9 \quad \overset{d}{\neg 10}$$

with reasons: 1_{α}, $\neg 3_{\beta}$, $\neg 4_{\gamma}$, $\neg 6_{\zeta}$, 7_{η}, 9_{κ} — decision level 4

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γ	$2 \vee \neg 4$
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$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ \overset{d}{\neg 10}\ \underset{\nu}{11}$$

Example

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$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ \overset{d}{\neg 10}\ \underset{\nu}{11}\ \overset{d}{12}$ decision level 5

Example

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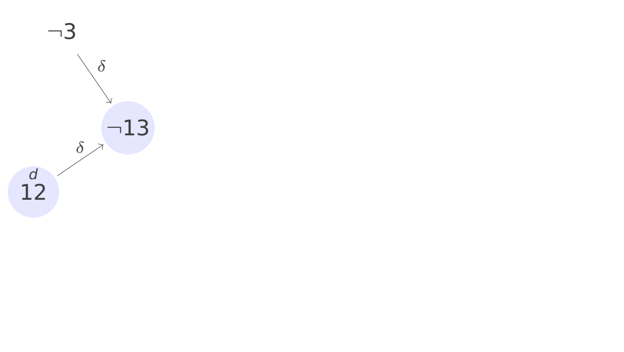
$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ \overset{d}{\neg 10}\ \underset{\nu}{11}\ \overset{d}{12}\ \underset{\delta}{\neg 13}$$



implication graph

Example

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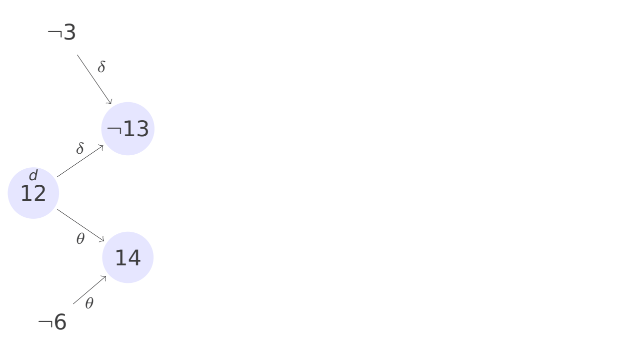

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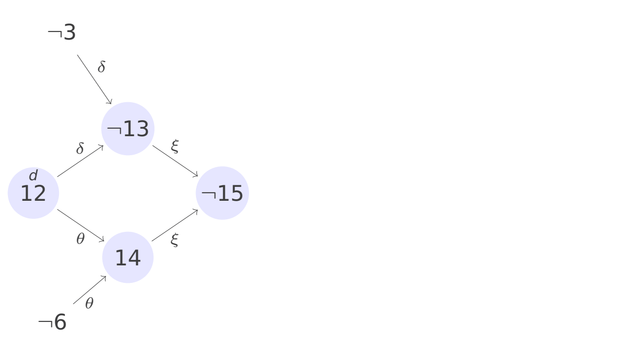
$$1_{\alpha}\ \neg 2^{d}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ 5^{d}\ \neg 6_{\zeta}\ 7_{\eta}\ 8^{d}\ 9_{\kappa}\ \neg 10^{d}\ 11_{\nu}\ 12^{d}\ \neg 13_{\delta}\ 14_{\theta}\ \neg 15_{\xi}$$



implication graph

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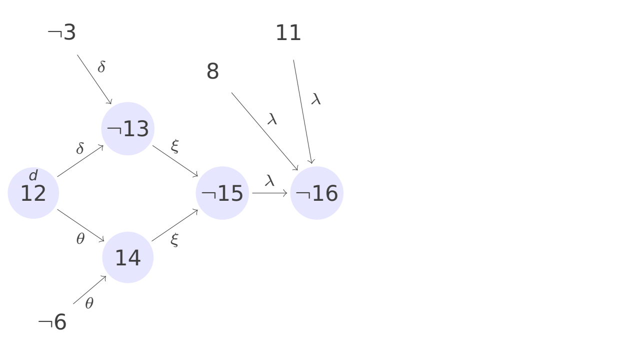

implication graph

Example

α 1

β $2 \vee \neg 3$

γ $2 \vee \neg 4$

δ $3 \vee \neg 12 \vee \neg 13$

ϵ $4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$

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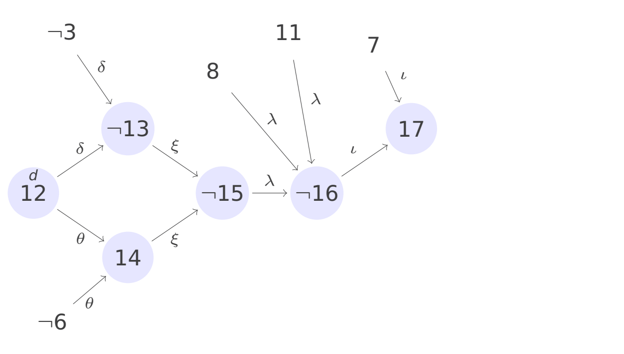
$$1_{\alpha}\ \neg 2^{d}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ 5^{d}\ \neg 6_{\zeta}\ 7_{\eta}\ 8^{d}\ 9_{\kappa}\ \neg 10^{d}\ 11_{\nu}\ 12^{d}\ \neg 13_{\delta}\ 14_{\theta}\ \neg 15_{\xi}\ \neg 16_{\lambda}\ 17_{\iota}$$



implication graph

Example

α	1
β	$2 \lor \neg 3$
γ	$2 \lor \neg 4$
δ	$3 \lor \neg 12 \lor \neg 13$
ϵ	$4 \lor \neg 17 \lor 18 \lor \neg 19 \lor 21$
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θ	$6 \lor \neg 12 \lor 14$
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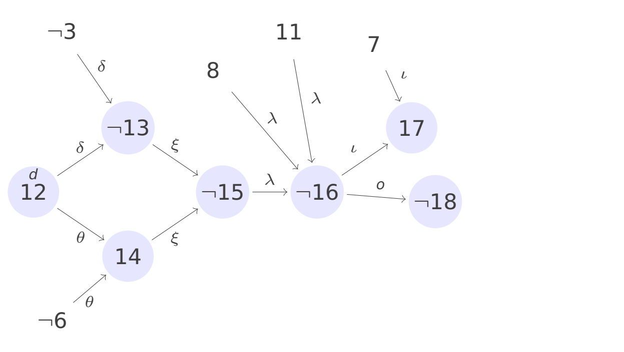

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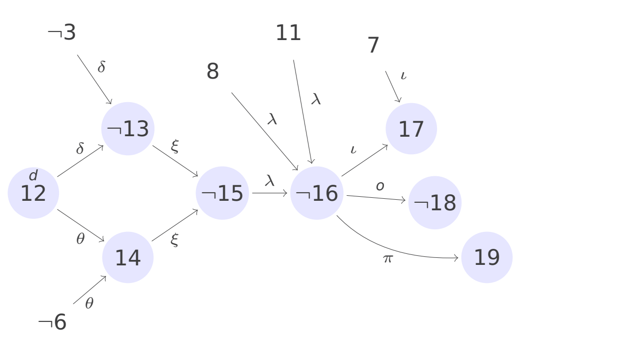
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$1_{\alpha}\ \neg 2^{d}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ 5^{d}\ \neg 6_{\zeta}\ 7_{\eta}\ 8^{d}\ 9_{\kappa}\ \neg 10^{d}\ 11_{\nu}\ 12^{d}\ \neg 13_{\delta}\ 14_{\theta}\ \neg 15_{\xi}\ \neg 16_{\lambda}\ 17_{\iota}\ \neg 18_{o}\ 19_{\pi}$$

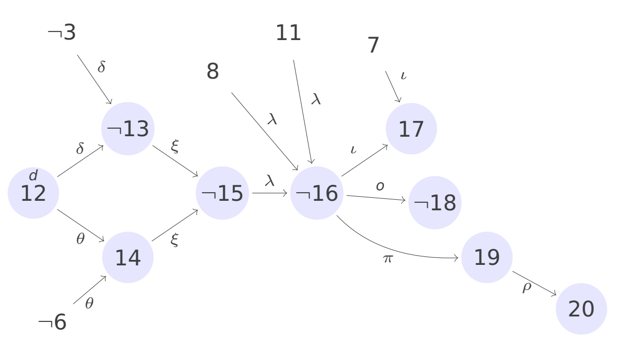


implication graph

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$1_\alpha\ \neg 2^d\ \neg 3_\beta\ \neg 4_\gamma\ 5^d\ \neg 6_\zeta\ 7_\eta\ 8^d\ 9_\kappa\ \neg 10^d\ 11_\nu\ 12^d\ \neg 13_\delta\ 14_\theta\ \neg 15_\xi\ \neg 16_\lambda\ 17_\iota\ \neg 18_o\ 19_\pi\ 20_\rho$$

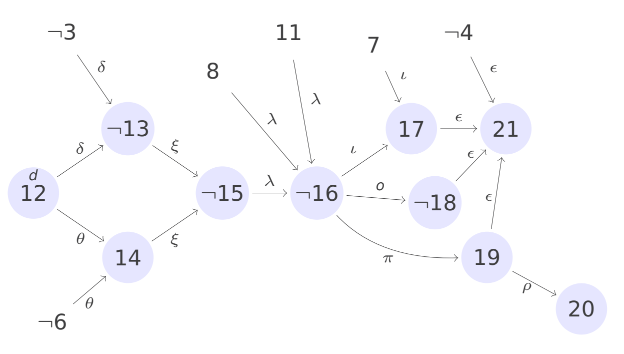


implication graph

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ \overset{d}{\neg 10}\ \underset{\nu}{11}\ \overset{d}{12}\ \underset{\delta}{\neg 13}\ \underset{\theta}{14}\ \underset{\xi}{\neg 15}\ \underset{\lambda}{\neg 16}\ \underset{\iota}{17}\ \underset{o}{\neg 18}\ \underset{\pi}{19}\ \underset{\rho}{20}\ \underset{\epsilon}{21}$$

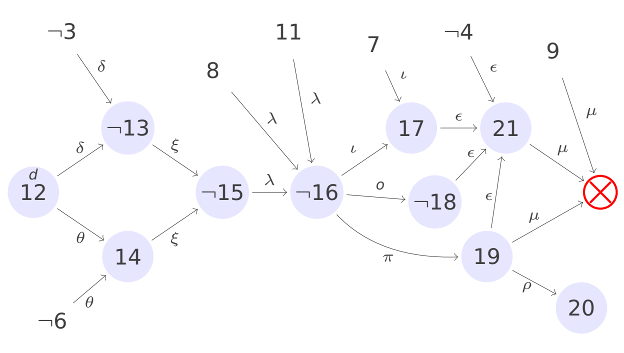


implication graph

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ \overset{d}{\neg 10}\ \underset{\nu}{11}\ \overset{d}{12}\ \underset{\delta}{\neg 13}\ \underset{\theta}{14}\ \underset{\xi}{\neg 15}\ \underset{\lambda}{\neg 16}\ \underset{\iota}{17}\ \underset{o}{\neg 18}\ \underset{\pi}{19}\ \underset{\rho}{20}\ \underset{\epsilon}{21}$

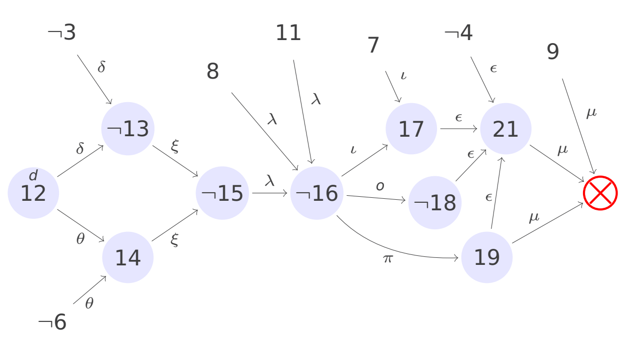


implication graph

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

	d			d			d		d		d									
1	$\neg 2$	$\neg 3$	$\neg 4$	5	$\neg 6$	7	8	9	$\neg 10$	11	12	$\neg 13$	14	$\neg 15$	$\neg 16$	17	$\neg 18$	19	20	21
α		β	γ		ζ	η		κ		ν		δ	θ	ξ	λ	ι	o	π	ρ	ϵ

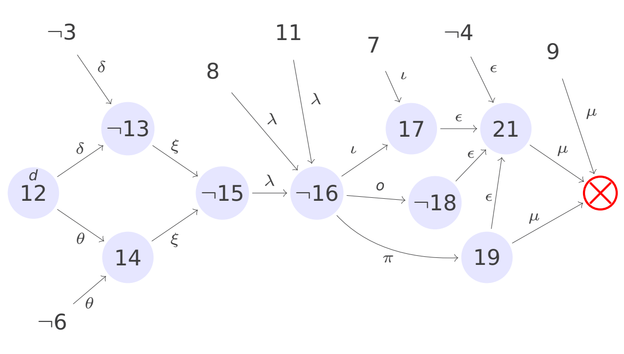


conflict graph

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ \overset{d}{\neg 10}\ \underset{\nu}{11}\ \overset{d}{12}\ \underset{\delta}{\neg 13}\ \underset{\theta}{14}\ \underset{\xi}{\neg 15}\ \underset{\lambda}{\neg 16}\ \underset{\iota}{17}\ \underset{o}{\neg 18}\ \underset{\pi}{19}\ \underset{\rho}{20}\ \underset{\epsilon}{21}$$

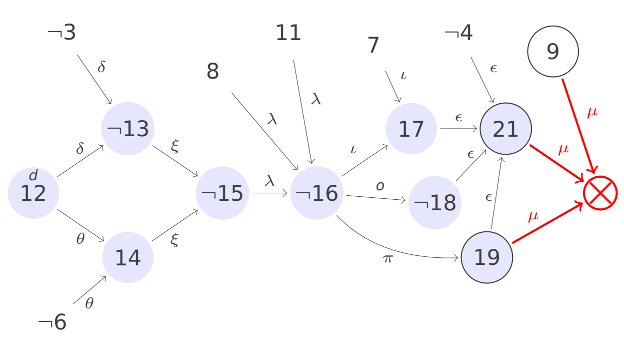


$\neg 9 \vee \neg 19 \vee \neg 21$ conflict clause (μ)

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$1_{\alpha}\ \neg 2^{d}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ 5^{d}\ \neg 6_{\zeta}\ 7_{\eta}\ 8^{d}\ 9_{\kappa}\ \neg 10^{d}\ 11_{\nu}\ 12^{d}\ \neg 13_{\delta}\ 14_{\theta}\ \neg 15_{\xi}\ \neg 16_{\lambda}\ 17_{\iota}\ \neg 18_{o}\ 19_{\pi}\ 20_{\rho}\ 21_{\epsilon}$$

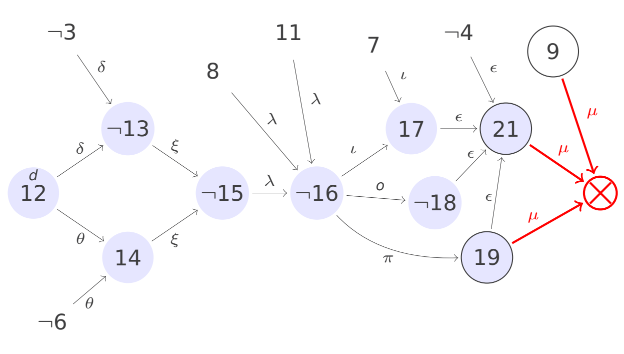


$\neg 9 \vee \neg 19 \vee \neg 21$ conflict clause (μ)

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$1_{\alpha}\ \overset{d}{\neg 2}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ \overset{d}{5}\ \neg 6_{\zeta}\ 7_{\eta}\ \overset{d}{8}\ 9_{\kappa}\ \overset{d}{\neg 10}\ 11_{\nu}\ \overset{d}{12}\ \neg 13_{\delta}\ 14_{\theta}\ \neg 15_{\xi}\ \neg 16_{\lambda}\ 17_{\iota}\ \neg 18_{o}\ 19_{\pi}\ 20_{\rho}\ 21_{\epsilon}$$



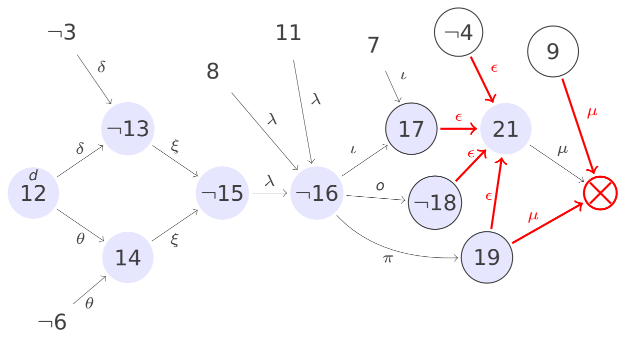
$\neg 9 \vee \neg 19 \vee \neg 21$ conflict clause (μ)

resolve with ϵ

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

	d			d			d		d		d									
1	$\neg 2$	$\neg 3$	$\neg 4$	5	$\neg 6$	7	8	9	$\neg 10$	11	12	$\neg 13$	14	$\neg 15$	$\neg 16$	17	$\neg 18$	19	20	21
α		β	γ		ζ	η		κ		ν		δ	θ	ξ	λ	ι	o	π	ρ	ϵ



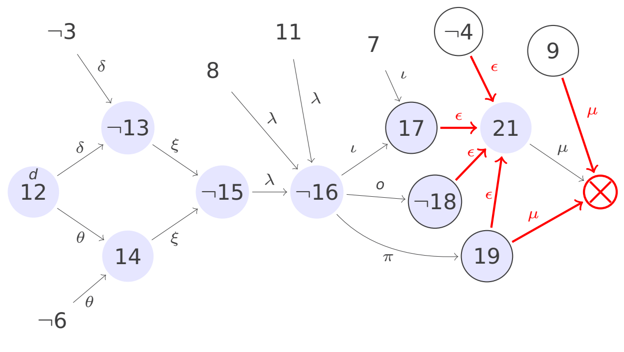
$\neg 9 \vee \neg 19 \vee \neg 21$ conflict clause (μ)

$4 \vee \neg 9 \vee \neg 17 \vee 18 \vee \neg 19$ resolve with ϵ

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ \overset{d}{\neg 10}\ \underset{\nu}{11}\ \overset{d}{12}\ \underset{\delta}{\neg 13}\ \underset{\theta}{14}\ \underset{\xi}{\neg 15}\ \underset{\lambda}{\neg 16}\ \underset{\iota}{17}\ \underset{o}{\neg 18}\ \underset{\pi}{19}\ \underset{\rho}{20}\ \underset{\epsilon}{21}$$



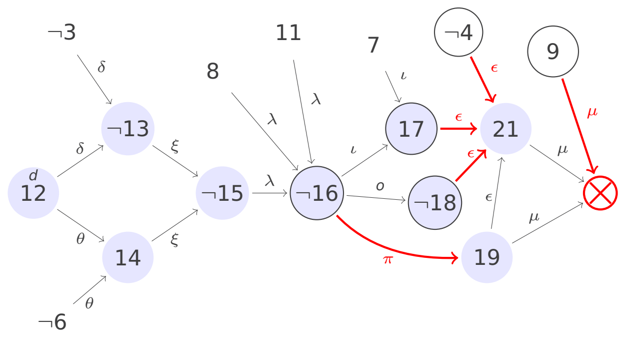
$\neg 9 \vee \neg 19 \vee \neg 21$ conflict clause (μ)

$4 \vee \neg 9 \vee \neg 17 \vee 18 \vee \neg 19$ resolve with ϵ, π

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

1	$\neg 2^d$	$\neg 3$	$\neg 4$	5^d	$\neg 6$	7	8^d	9	$\neg 10^d$	11	12^d	$\neg 13$	14	$\neg 15$	$\neg 16$	17	$\neg 18$	19	20	21
α		β	γ		ζ	η		κ		ν		δ	θ	ξ	λ	ι	o	π	ρ	ϵ



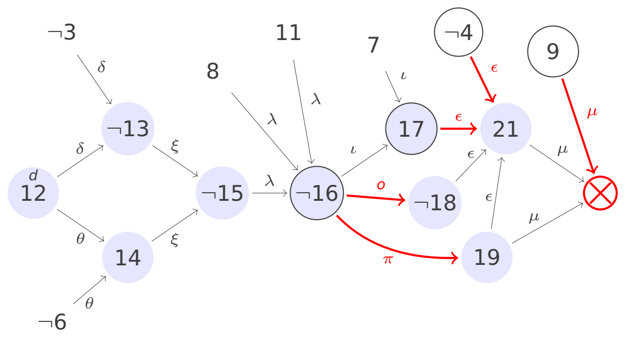
$\neg 9 \vee \neg 19 \vee \neg 21$ conflict clause (μ)

$4 \vee \neg 9 \vee 16 \vee \neg 17 \vee 18$ resolve with ϵ, π

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ \overset{d}{\neg 10}\ \underset{\nu}{11}\ \overset{d}{12}\ \underset{\delta}{\neg 13}\ \underset{\theta}{14}\ \underset{\xi}{\neg 15}\ \underset{\lambda}{\neg 16}\ \underset{\iota}{17}\ \underset{o}{\neg 18}\ \underset{\pi}{19}\ \underset{\rho}{20}\ \underset{\epsilon}{21}$$



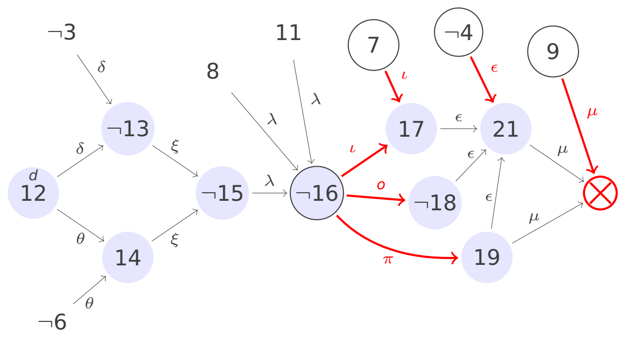
$\neg 9 \vee \neg 19 \vee \neg 21$ conflict clause (μ)

$4 \vee \neg 9 \vee 16 \vee \neg 17$ resolve with ϵ, π, o

Example

	d			d			d		d		d									
1	¬2	¬3	¬4	5	¬6	7	8	9	¬10	11	12	¬13	14	¬15	¬16	17	¬18	19	20	21
α		β	γ		ζ	η		κ		ν		δ	θ	ξ	λ	ι	o	π	ρ	ϵ

α	1
β	2 ∨ ¬3
γ	2 ∨ ¬4
δ	3 ∨ ¬12 ∨ ¬13
ϵ	4 ∨ ¬17 ∨ 18 ∨ ¬19 ∨ 21
ζ	¬5 ∨ ¬6
η	6 ∨ 7
θ	6 ∨ ¬12 ∨ 14
ι	¬7 ∨ 16 ∨ 17
κ	¬8 ∨ 9
λ	¬8 ∨ ¬11 ∨ 15 ∨ ¬16
μ	¬9 ∨ ¬19 ∨ ¬21
ν	10 ∨ 11
ξ	13 ∨ ¬14 ∨ ¬15
o	16 ∨ ¬18
π	16 ∨ 19
ρ	¬19 ∨ 20



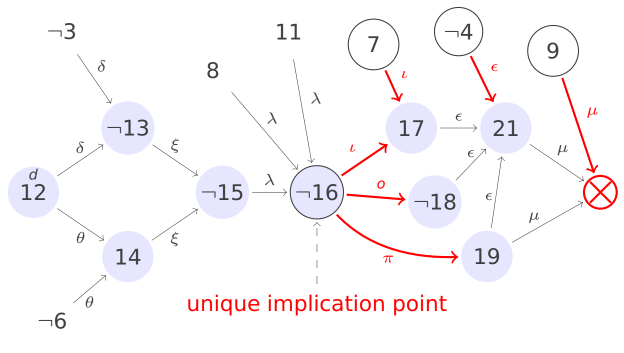
¬9 ∨ ¬19 ∨ ¬21 conflict clause (μ)

4 ∨ ¬7 ∨ ¬9 ∨ 16 resolve with ϵ, π, o, ι

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$1_{\alpha}\ \overset{d}{\neg 2}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ \overset{d}{5}\ \neg 6_{\zeta}\ 7_{\eta}\ \overset{d}{8}\ 9_{\kappa}\ \overset{d}{\neg 10}\ 11_{\nu}\ \overset{d}{12}\ \neg 13_{\delta}\ 14_{\theta}\ \neg 15_{\xi}\ \neg 16_{\lambda}\ 17_{\iota}\ \neg 18_{o}\ 19_{\pi}\ 20_{\rho}\ 21_{\epsilon}$$

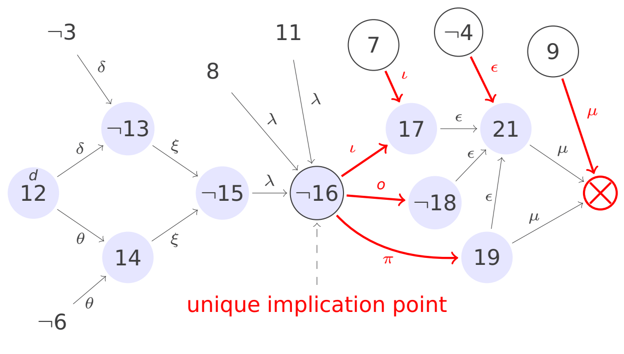


$\neg 9 \vee \neg 19 \vee \neg 21$	conflict clause (μ)
$4 \vee \neg 7 \vee \neg 9 \vee 16$	resolve with ϵ, π, o, ι

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ \overset{d}{\neg 10}\ \underset{\nu}{11}\ \overset{d}{12}\ \underset{\delta}{\neg 13}\ \underset{\theta}{14}\ \underset{\xi}{\neg 15}\ \underset{\lambda}{\neg 16}\ \underset{\iota}{17}\ \underset{o}{\neg 18}\ \underset{\pi}{19}\ \underset{\rho}{20}\ \underset{\epsilon}{21}$$



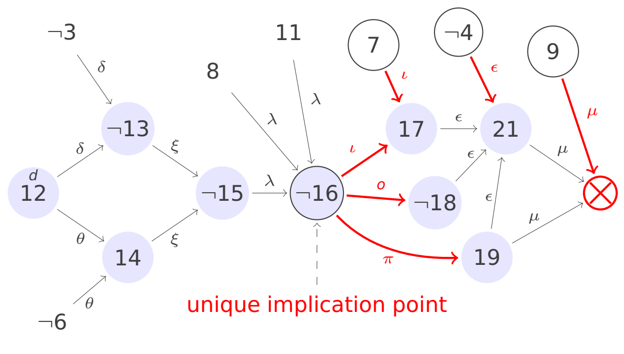
$\neg 9 \vee \neg 19 \vee \neg 21$	conflict clause (μ)
$4 \vee \neg 7 \vee \neg 9 \vee 16$	resolve with ϵ, π, o, ι

UIP: exactly one literal at current decision level

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$$1_{\alpha}\ \neg 2^{d}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ 5^{d}\ \neg 6_{\zeta}\ 7_{\eta}\ 8^{d}\ 9_{\kappa}\ \neg 10^{d}\ 11_{\nu}\ 12^{d}\ \neg 13_{\delta}\ 14_{\theta}\ \neg 15_{\xi}\ \neg 16_{\lambda}\ 17_{\iota}\ \neg 18_{o}\ 19_{\pi}\ 20_{\rho}\ 21_{\epsilon}$$



$\neg 9 \vee \neg 19 \vee \neg 21$ conflict clause (μ)

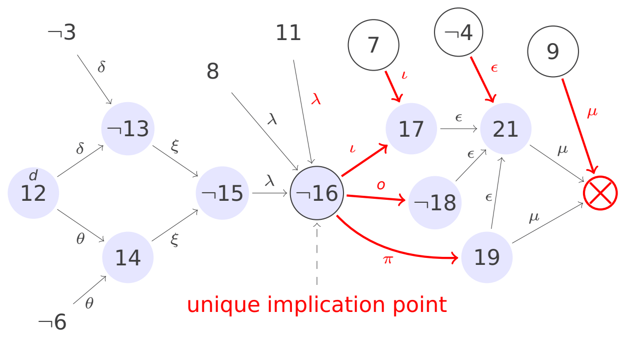
$4 \vee \neg 7 \vee \neg 9 \vee 16$ resolve with ϵ, π, o, ι

backjump clause

Example

α	1
β	$2 \vee \neg 3$
γ	$2 \vee \neg 4$
δ	$3 \vee \neg 12 \vee \neg 13$
ϵ	$4 \vee \neg 17 \vee 18 \vee \neg 19 \vee 21$
ζ	$\neg 5 \vee \neg 6$
η	$6 \vee 7$
θ	$6 \vee \neg 12 \vee 14$
ι	$\neg 7 \vee 16 \vee 17$
κ	$\neg 8 \vee 9$
λ	$\neg 8 \vee \neg 11 \vee 15 \vee \neg 16$
μ	$\neg 9 \vee \neg 19 \vee \neg 21$
ν	$10 \vee 11$
ξ	$13 \vee \neg 14 \vee \neg 15$
o	$16 \vee \neg 18$
π	$16 \vee 19$
ρ	$\neg 19 \vee 20$

$\underset{\alpha}{1}\ \overset{d}{\neg 2}\ \underset{\beta}{\neg 3}\ \underset{\gamma}{\neg 4}\ \overset{d}{5}\ \underset{\zeta}{\neg 6}\ \underset{\eta}{7}\ \overset{d}{8}\ \underset{\kappa}{9}\ 16$ backjump



$\neg 9 \vee \neg 19 \vee \neg 21$ conflict clause (μ)

$4 \vee \neg 7 \vee \neg 9 \vee 16$ resolve with ϵ, π, o, ι

backjump clause

Example

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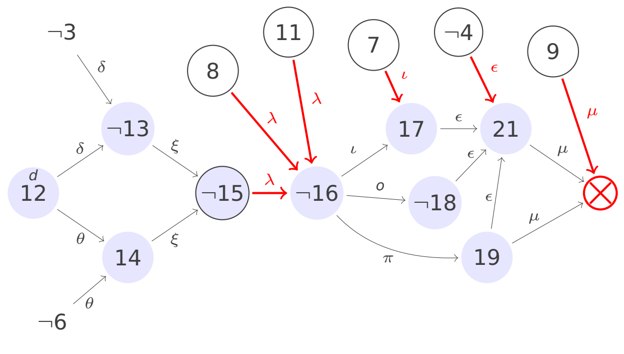
$$1_{\alpha}\ \overset{d}{\neg 2}\ \neg 3_{\beta}\ \neg 4_{\gamma}\ \overset{d}{5}\ \neg 6_{\zeta}\ 7_{\eta}\ \overset{d}{8}\ 9_{\kappa}\ \overset{d}{\neg 10}\ 11_{\nu}\ \overset{d}{12}\ \neg 13_{\delta}\ 14_{\theta}\ \neg 15_{\xi}\ \neg 16_{\lambda}\ 17_{\iota}\ \neg 18_{o}\ 19_{\pi}\ 20_{\rho}\ 21_{\epsilon}$$



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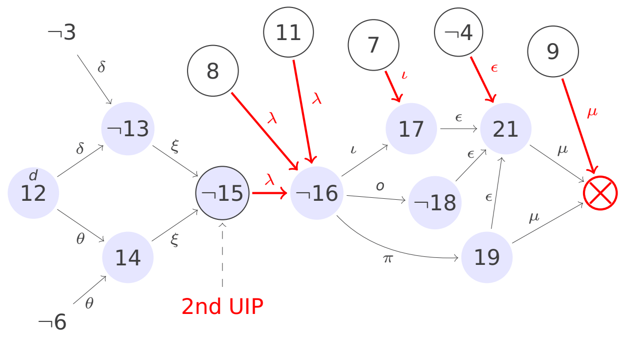
	d			d			d		d		d									
1	¬2	¬3	¬4	5	¬6	7	8	9	¬10	11	12	¬13	14	¬15	¬16	17	¬18	19	20	21
α		β	γ		ζ	η		κ		ν		δ	θ	ξ	λ	ι	o	π	ρ	ϵ



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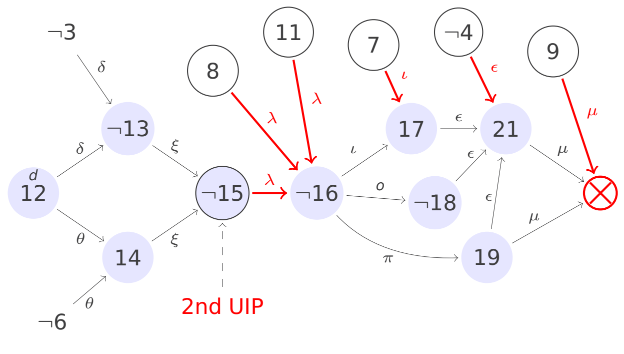

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$$1 \; \neg 2^d \; \neg 3 \; \neg 4 \; 5^d \; \neg 6 \; 7 \; 8^d \; 9 \; \neg 10^d \; 11 \; 12^d \; \neg 13 \; 14 \; \neg 15 \; \neg 16 \; 17 \; \neg 18 \; 19 \; 20 \; 21$$

1	$\neg 2$	$\neg 3$	$\neg 4$	5	$\neg 6$	7	8	9	$\neg 10$	11	12	$\neg 13$	14	$\neg 15$	$\neg 16$	17	$\neg 18$	19	20	21
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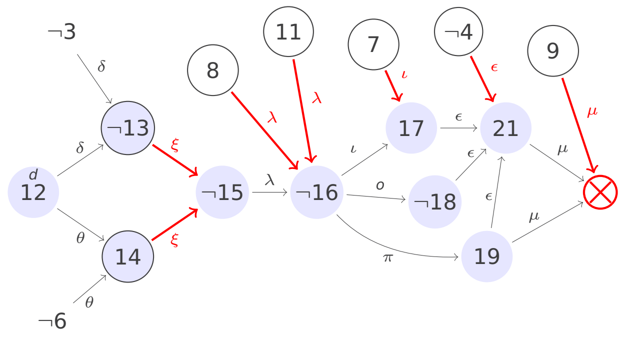

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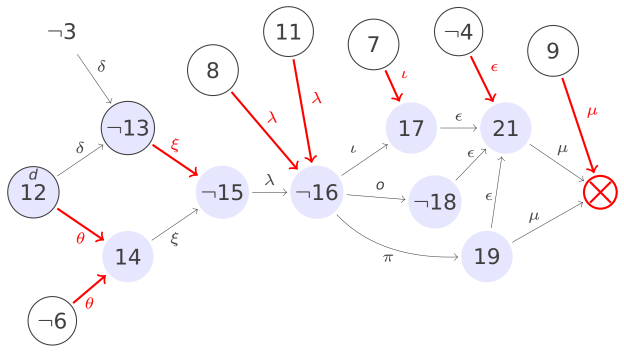

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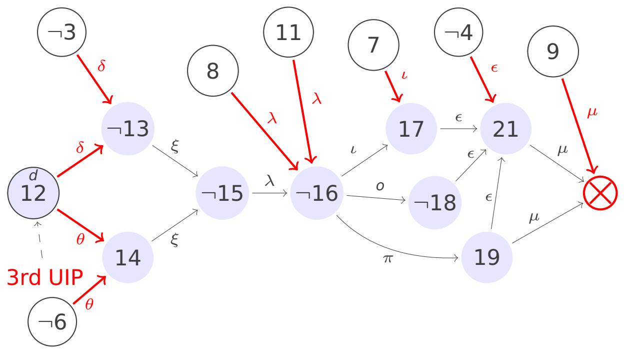

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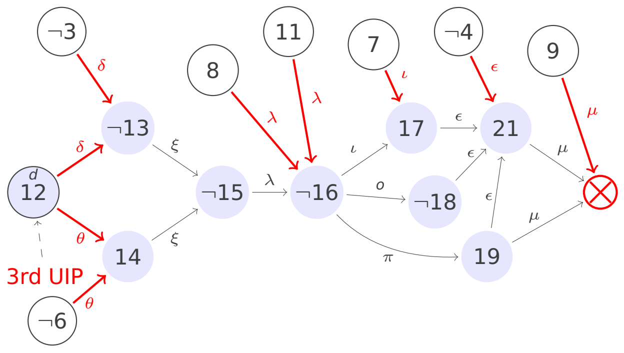

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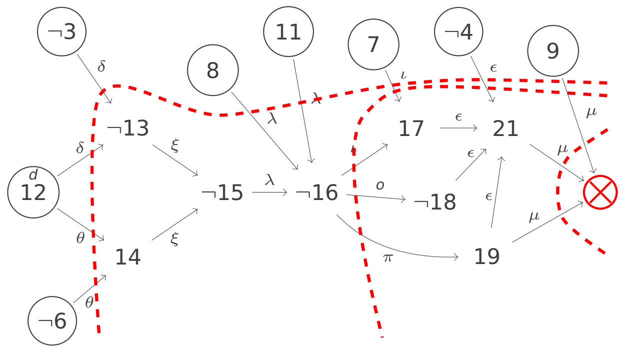

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- learn $\qquad M \parallel F \quad \Longrightarrow \quad M \parallel F, C$

  if $F \vDash C$ and each atom of C occurs in F or in M

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  - heuristics for selecting next decision literal
  - special data structures that allow for efficient unit propagation

Outline

1. Summary of Previous Lecture
2. Conflict Graphs
3. **NP-Completeness of SAT**
4. SAT Reductions
5. Further Reading

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Famous Open Problem

P $=$ NP ?

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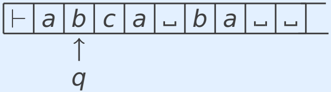

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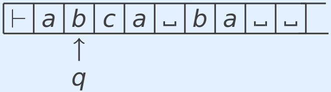

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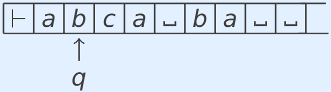

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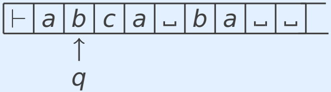

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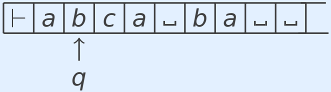

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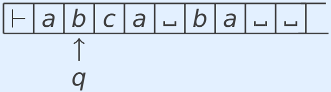

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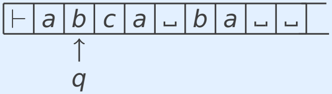

such that

$$\forall p \in F \quad \forall a \in \Gamma: \quad \Delta(p, a) = \varnothing$$

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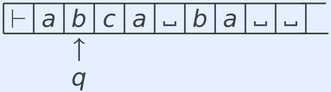
5. $\text{\textvisiblespace} \in \Gamma - \Sigma$: blank symbol
6. $\Delta : Q \times \Gamma \to 2^{Q \times \Gamma \times \{L,R\}}$: transition function
7. $s \in Q$: start state
8. $F \subseteq Q$: final states



such that

$$\forall p \in F \quad \forall a \in \Gamma: \quad \Delta(p, a) = \varnothing$$

$$\forall p \in Q \quad \forall (q, b, d) \in \Delta(p, \vdash): \quad b = \vdash \text{ and } d = R$$

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- assumption (WLOG): $\alpha \xrightarrow[M]{1} \alpha$ for every halting configuration α

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								$p(n)+1$-th configuration

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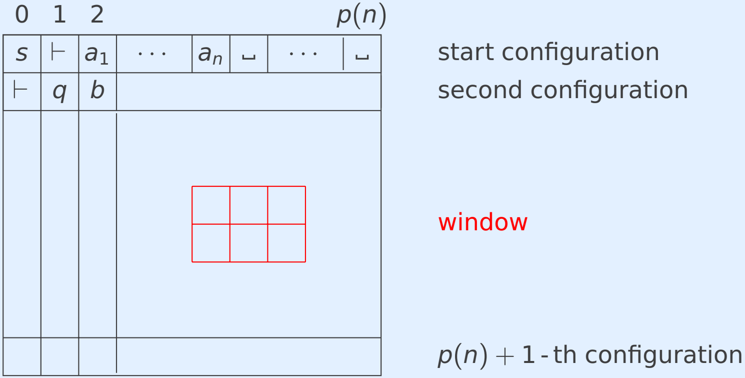

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$$\bigwedge_{i,j} \left[\bigvee_{a} \langle i, j, a\rangle \wedge \bigwedge_{a \neq b} \left(\neg\langle i, j, a\rangle \vee \neg\langle i, j, b\rangle\right) \right]$$

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$$\bigvee_{i,j}\bigvee_{q \in F}\langle i,j,q\rangle$$

Proof (cont'd)

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Example

suppose $\Delta(p,a) = \{(q,b,R)\}$ and $\Delta(p,b) = \{(p,c,L),(q,a,R)\}$

a	p	b
p	a	c

b	a	b
b	c	b

a	a	p
a	a	b

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~~b~~	~~c~~	~~b~~

a	a	p
a	a	b

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Outline

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- φ with clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$ over variables $\mathcal{V} = \{x_1, \ldots, x_n\}$
- bipartite graph $(\mathcal{C} \cup \mathcal{V}, \mathcal{E})$ with $\mathcal{E}$ containing edge $C_i - x_j$ if and only if C_i contains x_j or $\neg x_j$

Example

CNF $\varphi = \{\underbrace{\{x_1, x_2, x_3\}}_{C_1}, \underbrace{\{x_2, \neg x_3, x_4\}}_{C_2}, \underbrace{\{\neg x_1, \neg x_3, \neg x_4\}}_{C_3}\}$

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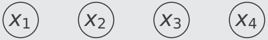

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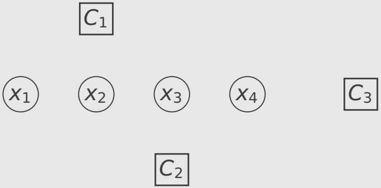

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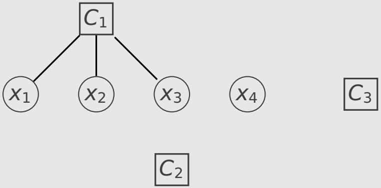

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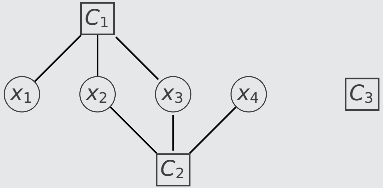

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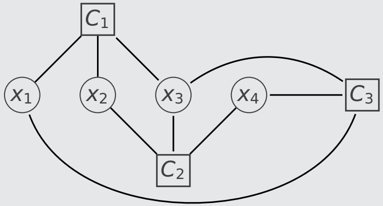

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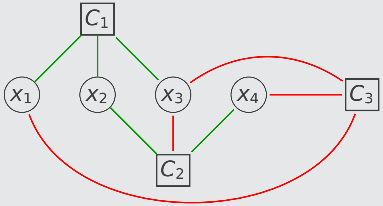

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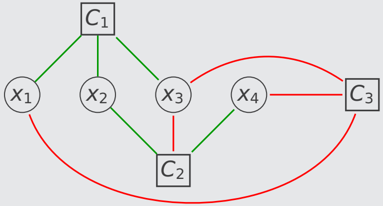

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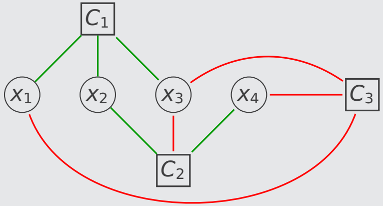

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Proof

NP-hardness follows by reduction from 3SAT

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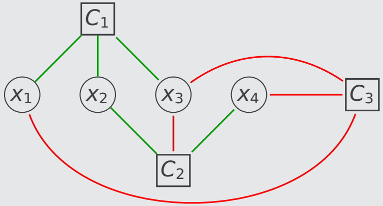

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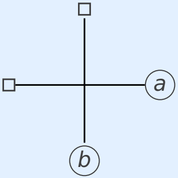
Remark

planar 3SAT is often used in reductions to show NP-hardness of particular problems

Main Idea (Crossover Gadget)



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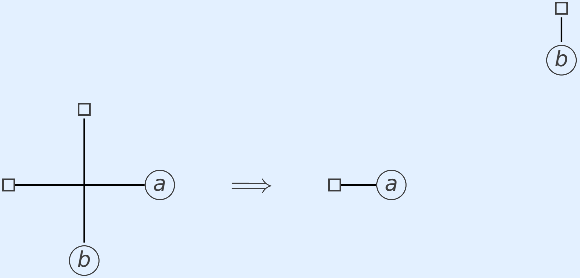
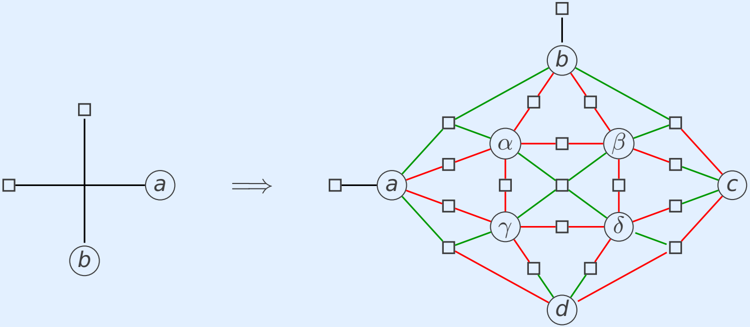
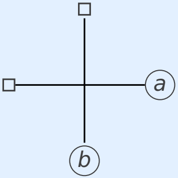


Main Idea (Crossover Gadget)



Main Idea (Crossover Gadget)



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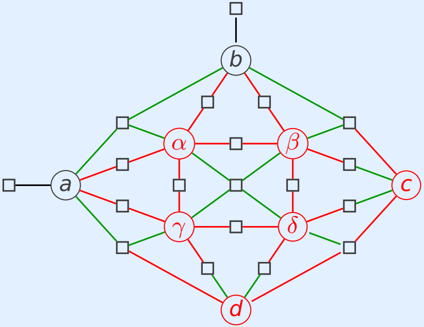
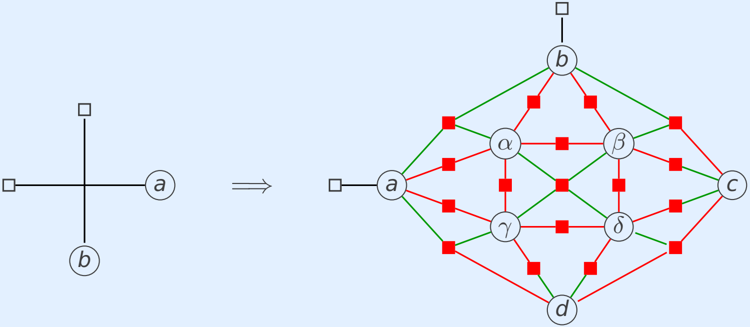


6 new variables

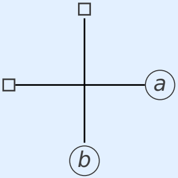
Main Idea (Crossover Gadget)



6 new variables

17 new clauses

Main Idea (Crossover Gadget)



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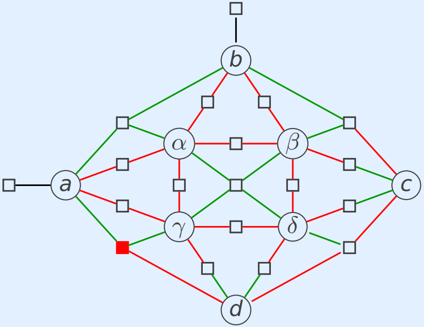
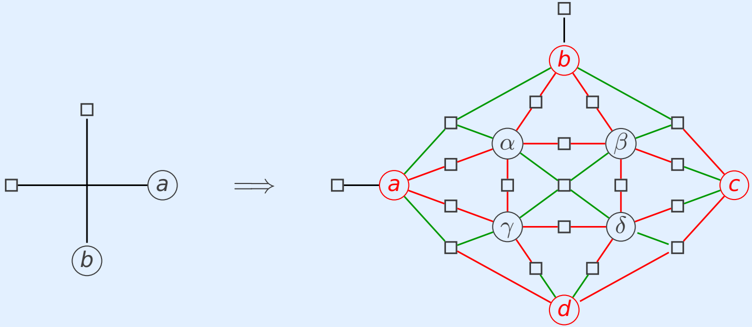


6 new variables

17 new clauses

■ $a \vee \gamma \vee \neg d$

Main Idea (Crossover Gadget)

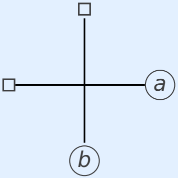


6 new variables

17 new clauses

claim: $c = a$ and $d = b$

Main Idea (Crossover Gadget)



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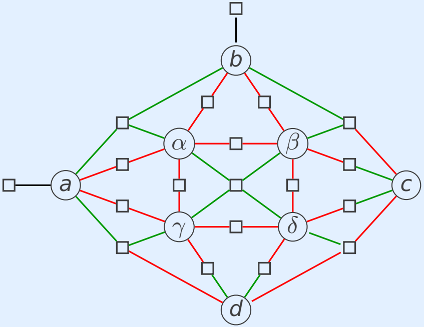


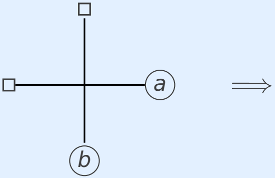
6 new variables

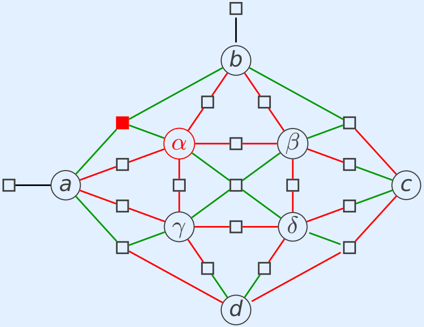
17 new clauses

claim: $c = a$ and $d = b$

a	b	α	β	γ	δ	c	d
0	0						
0	1						
1	0						
1	1						

Main Idea (Crossover Gadget)





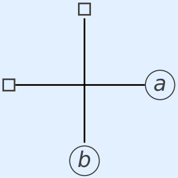
6 new variables

17 new clauses

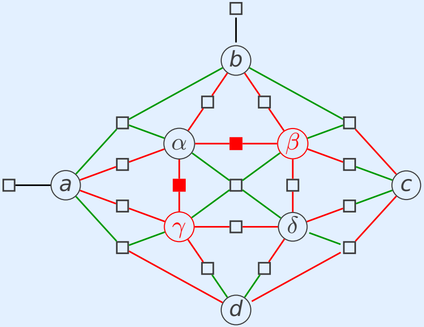
claim: $c = a$ and $d = b$

a	b	α	β	γ	δ	c	d
0	0	1					
0	1						
1	0						
1	1						

Main Idea (Crossover Gadget)



$\Longrightarrow$



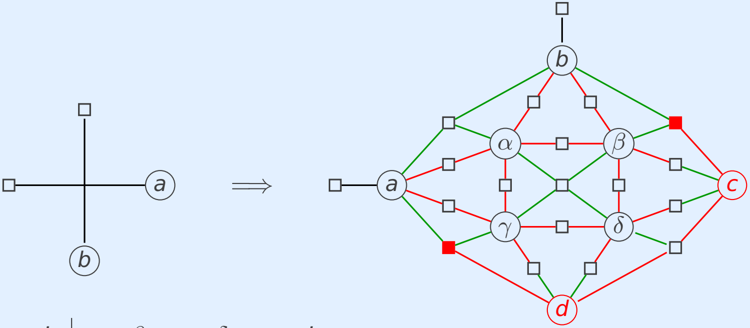
6 new variables

17 new clauses

claim: $c = a$ and $d = b$

a	b	α	β	γ	δ	c	d
0	0	1	0	0			
0	1						
1	0						
1	1						

Main Idea (Crossover Gadget)



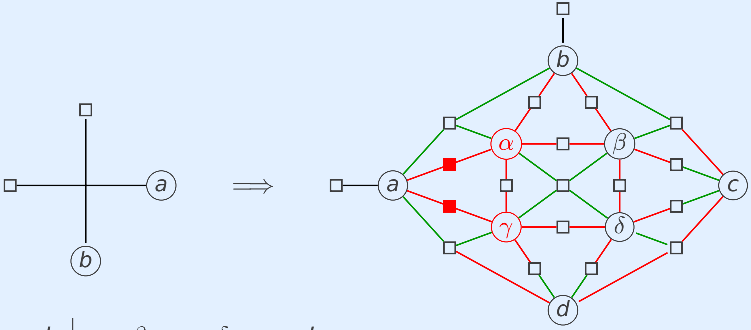
6 new variables

17 new clauses

claim: $c = a$ and $d = b$

a	b	α	β	γ	δ	c	d
0	0	1	0	0		0	0
0	1						
1	0						
1	1						

Main Idea (Crossover Gadget)



6 new variables

17 new clauses

claim: $c = a$ and $d = b$

a	b	α	β	γ	δ	c	d
0	0	1	0	0		0	0
0	1						
1	0	0		0			
1	1						

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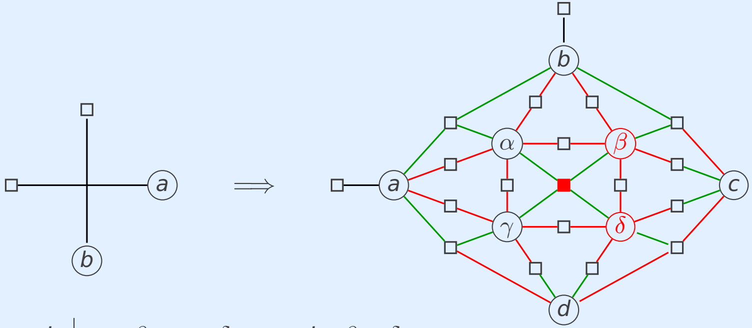


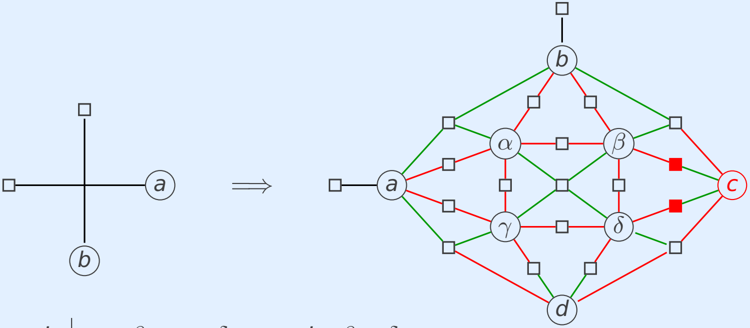
6 new variables

17 new clauses

claim: $c = a$ and $d = b$

a	b	α	β	γ	δ	c	d	$\beta \vee \delta$
0	0	1	0	0		0	0	
0	1							
1	0	0		0				1
1	1							

Main Idea (Crossover Gadget)



6 new variables

17 new clauses

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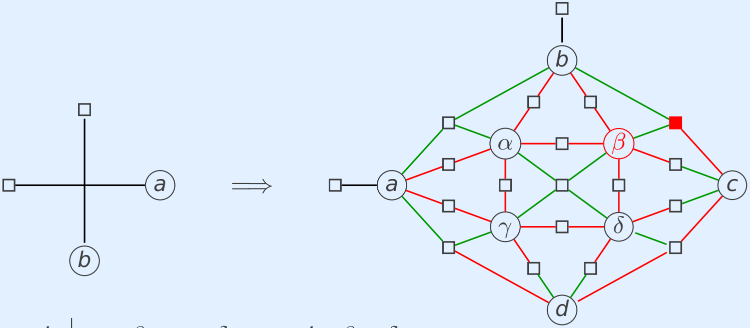
a	b	α	β	γ	δ	c	d	$\beta \vee \delta$
0	0	1	0	0		0	0	
0	1							
1	0	0		0		1		1
1	1							

Main Idea (Crossover Gadget)



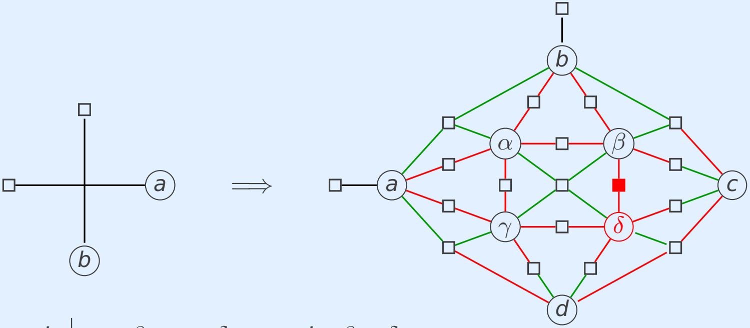
6 new variables

17 new clauses

claim: $c = a$ and $d = b$

a	b	α	β	γ	δ	c	d	$\beta \vee \delta$
0	0	1	0	0		0	0	
0	1							
1	0	0	1	0		1		1
1	1							

Main Idea (Crossover Gadget)



6 new variables

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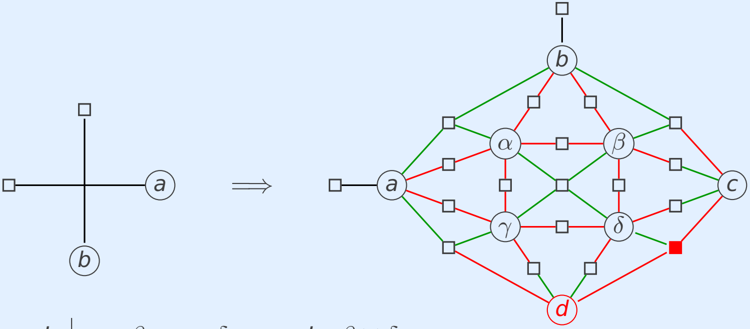
claim: $c = a$ and $d = b$

a	b	α	β	γ	δ	c	d	$\beta \vee \delta$
0	0	1	0	0		0	0	
0	1							
1	0	0	1	0	0	1		1
1	1							



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a	b	α	β	γ	δ	c	d	$\beta \vee \delta$
0	0	1	0	0		0	0	
0	1							
1	0	0	1	0	0	1	0	1
1	1							

Outline

Kröning and Strichmann

- Section 2.2

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Further Reading

- Stephen A. Cook
  The Complexity of Theorem-Proving Procedures
  Proc. 3rd ACM SToC, pp. 151 – 158, 1971

Kröning and Strichmann

- Section 2.2

Further Reading

- Stephen A. Cook
  The Complexity of Theorem-Proving Procedures
  Proc. 3rd ACM SToC, pp. 151 – 158, 1971

Further Viewing

- Erik Demaine
  Algorithmic Lower Bounds: Fun with Hardness Proofs
  MIT OpenCourseWare, 2014

Important Concepts

- 2SAT
- 3SAT
- conflict graph
- crossover gadget
- cut
- incidence graph
- learning
- NP
- NP-hard
- NP-complete
- P
- planar 3SAT
- reduction
- restart
- unique implication point