Constraint Solving

René Thiemann and Fabian Mitterwallner

based on a previous course by Aart Middeldorp

Motivation for extending SAT to first-order theories

predicates instead of propositional variables

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- equalities and disequalities over the reals

$$(x_1 = x_2 \vee x_1 = x_3) \wedge (x_1 = x_2 \vee x_1 = x_4) \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4$$

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- boolean combination of linear-arithmethic predicates

$$(x_1 + 2x_3 < 5) \vee \neg(x_3 \leqslant 1) \wedge (x_1 \geqslant x_3)$$

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- formula over arrays

$$(i = j \wedge a[j] = 1) \wedge \neg(a[i] = 1)$$

Outline

Definitions (Syntax)

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- notational conventions:
  - binding precedence $\neg \quad > \quad \wedge \quad > \quad \vee \quad > \quad \rightarrow \quad > \quad \forall, \exists$
  - omit outer parentheses, compress quantifiers: $\forall x\, y.\, \varphi$ instead of $\forall x. \forall y. \varphi$
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- sentence is formula without free variables

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- value $t^{\mathcal{M},l}$ of term t in model $\mathcal{M}$ relative to environment l is defined inductively:

$$t^{\mathcal{M},l} = \begin{cases} l(t) & \text{if } t \text{ is variable} \\ f^{\mathcal{M}}(t_1^{\mathcal{M},l}, \dots, t_n^{\mathcal{M},l}) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

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- environment for model $\mathcal{M} = (A, \{f^{\mathcal{M}}\}_{f \in \mathcal{F}}, \{P^{\mathcal{M}}\}_{P \in \mathcal{P}})$ is mapping I from variables to A
- value $t^{\mathcal{M},I}$ of term t in model $\mathcal{M}$ relative to environment I is defined inductively:

$$t^{\mathcal{M},I} = \begin{cases} I(t) & \text{if } t \text{ is variable} \\ f^{\mathcal{M}}(t_1^{\mathcal{M},I}, \dots, t_n^{\mathcal{M},I}) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

- given environment I, variable x and element $a \in A$, environment $I[x \mapsto a]$ is defined as

$$(I[x \mapsto a])(y) = \begin{cases} a & \text{if } y = x \\ I(y) & \text{if } y \neq x \end{cases}$$

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formula ψ

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- $\Gamma \vDash \psi$ (semantic entailment) if $\mathcal{M} \vDash_I \psi$ whenever $\mathcal{M} \vDash_I \varphi$ for all $\varphi \in \Gamma$, for all (appropriate) models $\mathcal{M}$ and environments I

Outline

First-Order Theories

formalize particular structures to enable reasoning about them

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Remark

axioms $\mathcal{A}$ provide meaning to symbols of Σ

Example (Addition on Natural Numbers: Presburger Arithmetic)

- signature: constant 0 unary function symbol s binary symbols $=$ $+$
- axioms (in addition to axioms for equality)
  - $\forall x.\, s(x) \neq 0$
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$>$ can be encoded: $x > y \iff \exists z.\, z \neq 0 \wedge x = y + z$

Example (Addition and Multiplication: Peano Arithmetic)

- signature: constant 0 unary function symbol s binary symbols $=$ $+$ $\times$
- axioms (PA)
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  - $\forall x.\, x \times 0 = 0$
  - $\forall x\, y.\, x \times s(y) = (x \times y) + x$

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- decidable if validity problem

  instance: sentence ψ over Σ

  question: $T \vDash \psi$?

  is decidable

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- universe: $\mathbb{N}$
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Proof Idea

sentence ψ encodes that ψ itself is unprovable in PA

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Satisfiability Modulo Theories (SMT)

theories are identified with their standard model

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fragment of theory $T = (\Sigma, \mathcal{A})$ is syntactically restricted subset of formulas over Σ

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Satisfiability Modulo Theories (SMT)

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- formulas are often restricted to quantifier-free fragment

Example (Binairo)

The objective of the number placement puzzle binairo is to fill a grid with 0's and 1's, where there is an equal number of 0's and 1's and no more than two consecutive 0's or 1's in each row and column. Additionally, identical rows and identical columns are forbidden. For instance, the binairo puzzle on the left has the solution on the right:

	0		0		0
	1			0	
0		1			
					1
		1		0	
				0	

1	0	1	0	1	0
1	1	0	1	0	0
0	0	1	0	1	1
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					1
	0				
1		1			1

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	0		0		0
	1			0	
0		1			
					1
		1		0	
				0	

1	2,6	3,6	4,6	5,6	6,6
1,5	2,5	3,5	0	5,5	6,5
1,4	2,4	3,4	4,4	5,4	1
1,3	2,3	3,3	4,3	5,3	6,3
1,2	0	3,2	4,2	5,2	6,2
1	2,1	1	4,1	5,1	1

1	0	1	0	1	0
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Remark

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SMT Encoding

$$\bigwedge_{i=1}^{6} \bigwedge_{j=1}^{6} \left(x_{i,j} = 0 \vee x_{i,j} = 1\right)$$

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SMT Encoding

$$\bigwedge_{i=1}^{6} \bigwedge_{j=1}^{6} \left(x_{i,j} = 0 \vee x_{i,j} = 1\right) \ \wedge \ \bigwedge_{i=1}^{6} \left(\sum_{j=1}^{6} x_{i,j} = 3\right) \ \wedge \ \bigwedge_{j=1}^{6} \left(\sum_{i=1}^{6} x_{i,j} = 3\right)$$

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SMT Encoding

$$\bigwedge_{i=1}^{6} \bigwedge_{j=1}^{6} (x_{i,j} = 0 \vee x_{i,j} = 1) \;\wedge\; \bigwedge_{i=1}^{6} \left(\sum_{j=1}^{6} x_{i,j} = 3 \right) \;\wedge\; \bigwedge_{j=1}^{6} \left(\sum_{i=1}^{6} x_{i,j} = 3 \right) \;\wedge$$

$$\bigwedge_{i=1}^{4} \bigwedge_{j=1}^{6} \left(\sum_{k=0}^{2} x_{i+k,j} = 1 \vee \sum_{k=0}^{2} x_{i+k,j} = 2 \right)$$

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$$\bigwedge_{i=1}^{4} \bigwedge_{j=1}^{6} \left(\sum_{k=0}^{2} x_{i+k,j} = 1 \vee \sum_{k=0}^{2} x_{i+k,j} = 2 \right) \;\wedge\; \bigwedge_{i=1}^{6} \bigwedge_{j=1}^{4} \left(\sum_{k=0}^{2} x_{i,j+k} = 1 \vee \sum_{k=0}^{2} x_{i,j+k} = 2 \right)$$

Remark

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SMT Encoding

$$\bigwedge_{i=1}^{6}\bigwedge_{j=1}^{6}(x_{i,j}=0 \vee x_{i,j}=1) \wedge \bigwedge_{i=1}^{6}\left(\sum_{j=1}^{6} x_{i,j}=3\right) \wedge \bigwedge_{j=1}^{6}\left(\sum_{i=1}^{6} x_{i,j}=3\right) \wedge$$

$$\bigwedge_{i=1}^{4}\bigwedge_{j=1}^{6}\left(\sum_{k=0}^{2} x_{i+k,j}=1 \vee \sum_{k=0}^{2} x_{i+k,j}=2\right) \wedge \bigwedge_{i=1}^{6}\bigwedge_{j=1}^{4}\left(\sum_{k=0}^{2} x_{i,j+k}=1 \vee \sum_{k=0}^{2} x_{i,j+k}=2\right) \wedge$$

$$\bigwedge_{i=1}^{5}\bigwedge_{k=i+1}^{6}\left(\bigvee_{j=1}^{6} x_{i,j}\neq x_{k,j}\right) \wedge \bigwedge_{j=1}^{5}\bigwedge_{k=j+1}^{6}\left(\bigvee_{i=1}^{6} x_{i,j}\neq x_{i,k}\right)$$

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$$x_{2,2}=0 \;\wedge\; x_{4,5}=0 \;\wedge\; x_{1,1}=1 \;\wedge\; x_{3,1}=1 \;\wedge\; x_{6,1}=1 \;\wedge\; x_{6,4}=1 \;\wedge\; x_{1,6}=1$$

Remark

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SMT Encoding (Linear Integer Arithmetic)

$$\bigwedge_{i=1}^{6} \bigwedge_{j=1}^{6} (x_{i,j} = 0 \vee x_{i,j} = 1) \;\wedge\; \bigwedge_{i=1}^{6} \left(\sum_{j=1}^{6} x_{i,j} = 3 \right) \;\wedge\; \bigwedge_{j=1}^{6} \left(\sum_{i=1}^{6} x_{i,j} = 3 \right) \;\wedge$$

$$\bigwedge_{i=1}^{4} \bigwedge_{j=1}^{6} \left(\sum_{k=0}^{2} x_{i+k,j} = 1 \vee \sum_{k=0}^{2} x_{i+k,j} = 2 \right) \;\wedge\; \bigwedge_{i=1}^{6} \bigwedge_{j=1}^{4} \left(\sum_{k=0}^{2} x_{i,j+k} = 1 \vee \sum_{k=0}^{2} x_{i,j+k} = 2 \right) \;\wedge$$

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$$x_{2,2} = 0 \;\wedge\; x_{4,5} = 0 \;\wedge\; x_{1,1} = 1 \;\wedge\; x_{3,1} = 1 \;\wedge\; x_{6,1} = 1 \;\wedge\; x_{6,4} = 1 \;\wedge\; x_{1,6} = 1$$

SMT-LIB 2 Format

```
(declare-const x11 Int)  ...  (declare-const x66 Int)
```

SMT-LIB 2 Format

```
(declare-const x11 Int)  ...  (declare-const x66 Int)
(assert (or (= x11 0) (= x11 1)))  ...  (assert (or (= x66 0) (= x66 1)))
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(declare-const x11 Int)  ...  (declare-const x66 Int)
(assert (or (= x11 0) (= x11 1)))  ...  (assert (or (= x66 0) (= x66 1)))
(assert (= (+ x11 x12 x13 x14 x15 x16) 3))
...
(assert (= (+ x16 x26 x36 x46 x56 x66) 3))
```

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(declare-const x11 Int)  ...  (declare-const x66 Int)
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...
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(assert (or (= (+ x11 x21 x31) 1) (= (+ x11 x21 x31) 2)))
...
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...
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            (not (= x14 x24)) (not (= x15 x25)) (not (= x16 x26))))
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Links

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- Z3 bindings for Haskell
- SBV: SMT Based Verification in Haskell

Outline

SMT Problem

decide satisfiability of formulas in

propositional logic + domain-specific background theories

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SMT Solving: Lazy Approach

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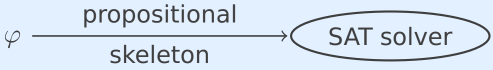
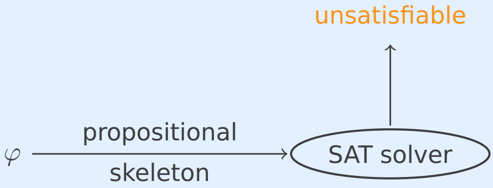
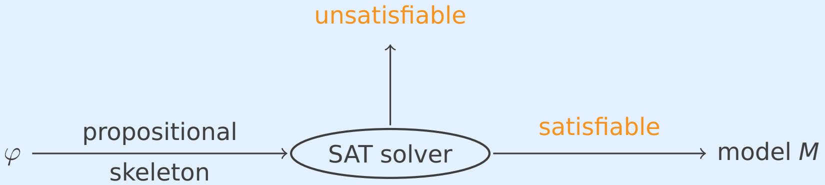


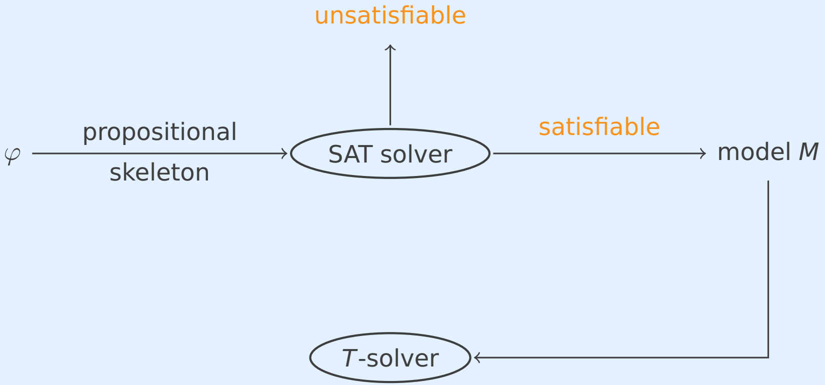
SMT Solving: Lazy Approach



SMT Solving: Lazy Approach



SMT Solving: Lazy Approach



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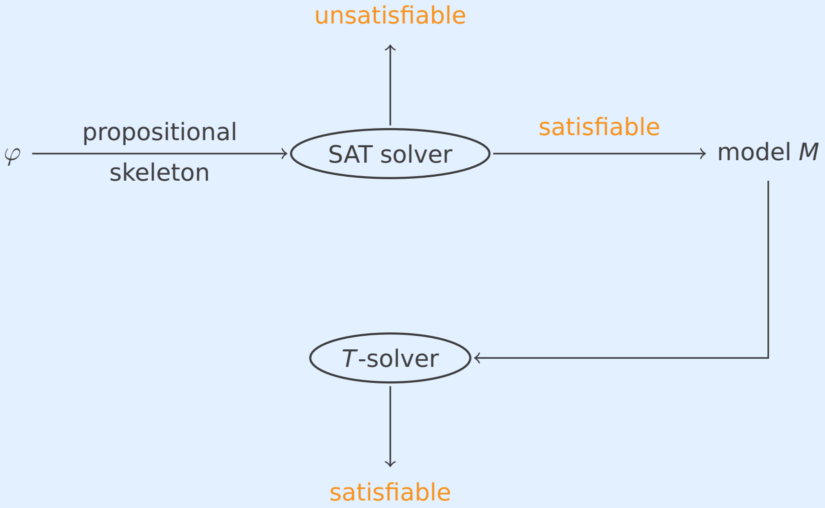

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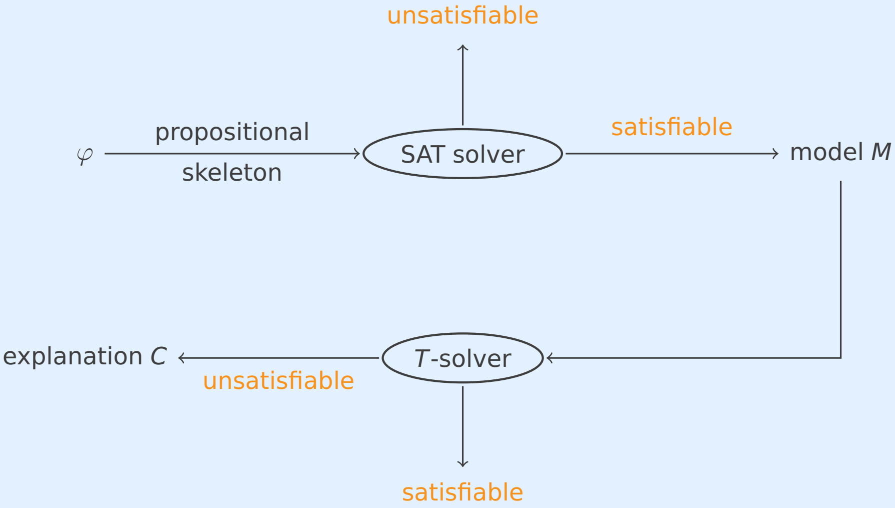

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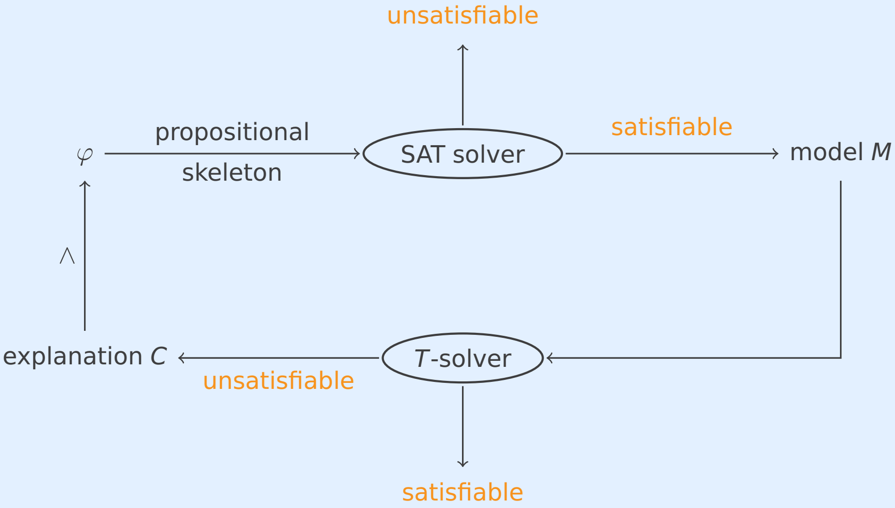

Example

formula $x = 1 \wedge (\neg(y = 1) \vee \neg(x + 2y = 3)) \wedge x + y = 2$

Example

formula $\underbrace{x = 1}_{a} \wedge (\neg\underbrace{(y = 1)}_{b} \vee \neg\underbrace{(x + 2y = 3)}_{c}) \wedge \underbrace{x + y = 2}_{d}$

- input to SAT solver (propositional skeleton)

$$a \wedge (\neg b \vee \neg c) \wedge d$$

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- input to LIA solver

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blocking clause

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Example

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Example

formula $\underset{a}{x = 1} \;\wedge\; (\neg\underset{b}{(y = 1)} \;\vee\; \neg\underset{c}{(x + 2y = 3)}) \;\wedge\; \underset{d}{x + y = 2}$

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Example

formula $\underbrace{x = 1}_{a} \wedge (\neg\underbrace{(y = 1)}_{b} \vee \neg\underbrace{(x + 2y = 3)}_{c}) \wedge \underbrace{x + y = 2}_{d}$ is unsatisfiable

- input to SAT solver

$$a \wedge (\neg b \vee \neg c) \wedge d \wedge (\neg a \vee b \vee \neg d) \wedge (\neg a \vee \neg b \vee c)$$

- SAT solver reports unsatisfiable

Outline

most state-of-the-art SMT solvers use DPLL(T)

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general framework for lazy SMT solving with theory propagation

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Definitions

first-order theory T, formulas F and G, list of literals M

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- T-backjump $\qquad M \overset{d}{l} N \parallel F, C \implies M\, l' \parallel F, C$

  if $M \overset{d}{l} N \vDash \neg C$ and $\exists$ clause $C' \vee l'$ such that

  - $F, C \vDash_T C' \vee l'$ and $M \vDash \neg C'$
  - l' is undefined in M and l' or l'^c occurs in F or in $M \overset{d}{l} N$

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- T-forget $$M \parallel F, C \quad \Longrightarrow \quad M \parallel F$$

  if $F \vDash_T C$

Definition

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  - l' is undefined in M and l' or l'^c occurs in F or in $M\ l^d\ N$

- T-learn $\qquad M \parallel F \implies M \parallel F, C$

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- T-forget $\qquad M \parallel F, C \implies M \parallel F$

  if $F \vDash_T C$

- T-propagate $\qquad M \parallel F \implies M\ l \parallel F$

  if $M \vDash_T l$, l is undefined in M, and l or l^c occurs in F

Example

(EUF) formula $\underbrace{g(a) = c}_{1} \wedge (\neg \underbrace{(f(g(a)) = f(c))}_{2} \vee \underbrace{g(a) = d}_{3}) \wedge \neg \underbrace{(c = d)}_{4}$

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$$\Longrightarrow \quad 1 \ \parallel\ 1,\ \neg 2 \vee 3,\ \neg 4$$

unit propagate

Example

(EUF) formula $\underbrace{g(a) = c}_{1} \land (\neg \underbrace{(f(g(a)) = f(c))}_{2} \lor \underbrace{g(a) = d}_{3}) \land \neg \underbrace{(c = d)}_{4}$

$$\| \; 1, \; \neg 2 \lor 3, \; \neg 4$$

$$\Longrightarrow \quad 1 \; \| \; 1, \; \neg 2 \lor 3, \; \neg 4 \qquad \text{unit propagate}$$

$$\Longrightarrow \quad 1 \; \neg 4 \; \| \; 1, \; \neg 2 \lor 3, \; \neg 4 \qquad \text{unit propagate}$$

Example

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$$\begin{array}{lrll}
 & & \| \ 1,\ \neg 2 \vee 3,\ \neg 4 & \\
\Longrightarrow & 1 & \| \ 1,\ \neg 2 \vee 3,\ \neg 4 & \text{unit propagate} \\
\Longrightarrow & 1\ \neg 4 & \| \ 1,\ \neg 2 \vee 3,\ \neg 4 & \text{unit propagate} \\
\Longrightarrow & 1\ \neg 4\ \neg 2^{d} & \| \ 1,\ \neg 2 \vee 3,\ \neg 4 & \text{decide}
\end{array}$$

Example

(EUF) formula $\underbrace{g(a) = c}_{1} \wedge (\neg \underbrace{(f(g(a)) = f(c))}_{2} \vee \underbrace{g(a) = d}_{3}) \wedge \neg \underbrace{(c = d)}_{4}$

		$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	
$\Longrightarrow$	1	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	unit propagate
$\Longrightarrow$	$1\ \neg 4$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	unit propagate
$\Longrightarrow$	$1\ \neg 4\ \neg 2^{d}$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	decide
$\Longrightarrow$	$1\ \neg 4\ \neg 2^{d}$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4$	T-learn

Example

(EUF) formula $\underset{1}{g(a) = c} \wedge (\neg(\underset{2}{f(g(a)) = f(c)}) \vee \underset{3}{g(a) = d}) \wedge \neg(\underset{4}{c = d})$

		$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	
$\Longrightarrow$	1	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	unit propagate
$\Longrightarrow$	$1\ \neg 4$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	unit propagate
$\Longrightarrow$	$1\ \neg 4\ \neg 2^d$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	decide
$\Longrightarrow$	$1\ \neg 4\ \neg 2^d$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4$	T-learn
$\Longrightarrow$	$1\ \neg 4\ 2$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4$	T-backjump

Example

(EUF) formula $\underbrace{g(a) = c}_{1} \wedge (\neg(\underbrace{f(g(a)) = f(c)}_{2}) \vee \underbrace{g(a) = d}_{3}) \wedge \neg(\underbrace{c = d}_{4})$

		$\Vert$	$1,\ \neg 2 \vee 3,\ \neg 4$	
$\Longrightarrow$	1	$\Vert$	$1,\ \neg 2 \vee 3,\ \neg 4$	unit propagate
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$\Longrightarrow$	$1\ \neg 4\ \neg 2^{d}$	$\Vert$	$1,\ \neg 2 \vee 3,\ \neg 4$	decide
$\Longrightarrow$	$1\ \neg 4\ \neg 2^{d}$	$\Vert$	$1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4$	T-learn
$\Longrightarrow$	$1\ \neg 4\ 2$	$\Vert$	$1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4$	T-backjump
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Example

(EUF) formula $\underbrace{g(a) = c}_{1} \wedge (\neg(\underbrace{f(g(a)) = f(c)}_{2}) \vee \underbrace{g(a) = d}_{3}) \wedge \neg(\underbrace{c = d}_{4})$

		$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	
$\Longrightarrow$	1	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	unit propagate
$\Longrightarrow$	$1\ \neg 4$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	unit propagate
$\Longrightarrow$	$1\ \neg 4\ \neg 2^{d}$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4$	decide
$\Longrightarrow$	$1\ \neg 4\ \neg 2^{d}$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4$	T-learn
$\Longrightarrow$	$1\ \neg 4\ 2$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4$	T-backjump
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$\Longrightarrow$	$1\ \neg 4\ 2\ 3$	$\parallel$	$1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4,\ \neg 1 \vee \neg 2 \vee \neg 3 \vee 4$	T-learn

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$$
\begin{array}{llll}
 & & \| \; 1,\ \neg 2 \vee 3,\ \neg 4 & \\
\Longrightarrow & 1 & \| \; 1,\ \neg 2 \vee 3,\ \neg 4 & \text{unit propagate} \\
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\Longrightarrow & 1\ \neg 4\ 2 & \| \; 1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4 & T\text{-backjump} \\
\Longrightarrow & 1\ \neg 4\ 2\ 3 & \| \; 1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4 & \text{unit propagate} \\
\Longrightarrow & 1\ \neg 4\ 2\ 3 & \| \; 1,\ \neg 2 \vee 3,\ \neg 4,\ \neg 1 \vee 2 \vee 4,\ \neg 1 \vee \neg 2 \vee \neg 3 \vee 4 & T\text{-learn} \\
\Longrightarrow & \text{fail-state} & & \text{fail}
\end{array}
$$

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check T-consistency of M with T-solver

1. if M is T-consistent then F is T-satisfiable
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Improvements

1. apply fail or T-backjump after T-learn (instead of restart)

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2. if M is not T-consistent then $F \models_T \neg(l_1 \wedge \cdots \wedge l_k)$ for some literals $l_1, \ldots, l_k$ in M

   add blocking clause $\neg l_1 \vee \cdots \vee \neg l_k$ by T-learn and apply restart

Improvements

1. apply fail or T-backjump after T-learn (instead of restart)
2. check T-consistency of M or apply T-propagate before decide

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lazy SMT approach is modeled in DPLL(T) as follows:

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   add blocking clause $\neg l_1 \vee \cdots \vee \neg l_k$ by T-learn and apply restart

Improvements

1. apply fail or T-backjump after T-learn (instead of restart)
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3. find small unsatisfiable cores to minimize k in blocking clauses

Outline

Kröning and Strichmann

- Section 1.4
- Chapter 3

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Further Reading

- Robert Nieuwenhuis, Albert Oliveras, and Cesare Tinelli
  Solving SAT and SAT Modulo Theories: From an Abstract Davis–Putnam–Logemann–Loveland Procedure to DPLL(T)
  Journal of the ACM 53(6), pp. 937 – 977, 2006

Important Concepts

- $\equiv_T$
- $\vDash_T$
- blocking clause
- complete theory
- conjunctive fragment
- consistent theory
- decidable theory
- DPLL(T)
- first-order formula
- fragment
- model
- Peano arithmetic
- propositional skeleton
- quantifier-free fragment
- sentence
- standard model
- T-backjump
- T-consistency
- T-learn
- T-propagate
- T-satisfiability
- T-solver