universität innsbruck

SS 2024 lecture 5

Constraint Solving

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based on a previous course by Aart Middeldorp

Outline

1. **Summary of Previous Lecture**
2. **Equality Logic**
3. **Equality Logic with Uninterpreted Functions**
4. **EUF**
5. **Congruence Closure**
6. **Further Reading**

SMT Problem

decide satisfiability of (quantifier-free) formulas in

propositional logic + domain-specific background theories (axiomatic or concrete model)

Terminology

theory solver for T (T-solver) is procedure for deciding T-satisfiability of conjunction of quantifier-free literals

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Remark

- SMT solvers often use DPLL(T) framework
- DPLL(T): combine DPLL-based SAT-solver with T-solver; the latter is used for
  - T-consistency checks – find model w.r.t. theory or generate blocking clause
  - T-propagation – find implied literals
  - basic implementation of T-propagation: $M \models_T l$ if $M \wedge \neg l$ is unsatisfiable

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$$y = z \ \wedge\ x = z \quad \vee \quad x \neq z \ \wedge\ x = y$$

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small model property: satisfiable formula φ with n variables has model with domain $\{1, \ldots, n\}$

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Consequence

from now on consider equality logic without constants

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satisfiability problem for equality logic is NP-complete

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  - equality logic formula ψ is obtained from φ by replacing every p_i with $x_i = y_i$
  - φ is satisfiable $\iff$ ψ is satisfiable

Satisfiability Procedure for Conjunctive Fragment of Equality Logic

easy but important case: conjunction of equalities and disequalities φ

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Examples

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$$x_1 = x_2 \;\vee\; (x_3 = x_5 \;\wedge\; x_2 \neq x_5 \;\wedge\; x_1 = x_5)$$

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1. define equivalence class for each variable in φ

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T-solver for equality logic

conjunction φ of equality logic literals over set of variables V

Definitions

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Lemma

φ is satisfiable $\iff$ $G_=(\varphi)$ contains no simple contradictory cycles

Example

formula φ

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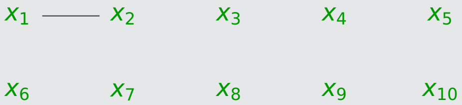

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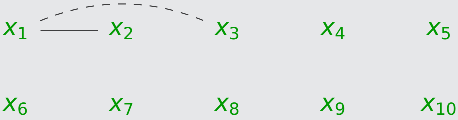

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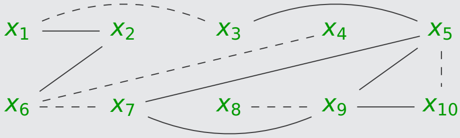

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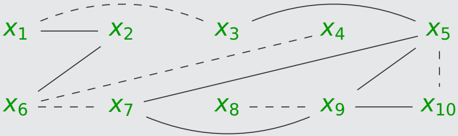

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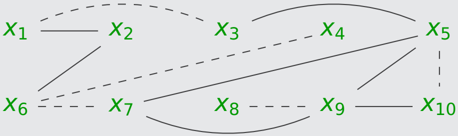

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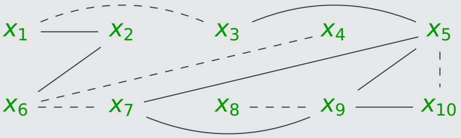

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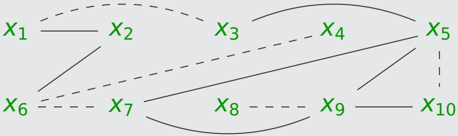

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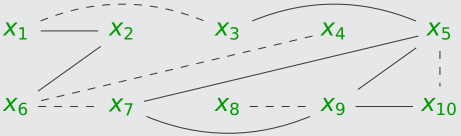
Example

formula φ

$$x_1 = x_2 \;\wedge\; x_1 \neq x_3 \;\wedge\; x_3 = x_5 \;\wedge\; x_4 \neq x_6 \;\wedge\; x_6 \neq x_7 \;\wedge\; x_5 = x_9 \;\wedge$$
$$x_2 = x_6 \;\wedge\; x_5 = x_7 \;\wedge\; x_8 \neq x_9 \;\wedge\; x_9 = x_{10} \;\wedge\; x_7 = x_9 \;\wedge\; x_5 \neq x_{10}$$

- equality graph $G_=(\varphi)$



- contradictory cycles

simple

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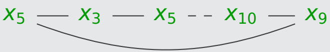

- φ is unsatisfiable

Outline

1. Summary of Previous Lecture
2. Equality Logic
3. **Equality Logic with Uninterpreted Functions**
4. EUF
5. Congruence Closure
6. Further Reading

Aim

- further increase expressivity of logic
- one solution: add uninterpreted functions

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  - function congruence (for every n-ary function symbol f)

$$\forall x_1 \dots x_n\, y_1 \dots y_n.\; x_1 = y_1 \land \cdots \land x_n = y_n \;\rightarrow\; f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$$

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$$\forall x_1 \dots x_n\, y_1 \dots y_n.\, x_1 = y_1 \wedge \cdots \wedge x_n = y_n \rightarrow (P(x_1, \dots, x_n) \leftrightarrow P(y_1, \dots, y_n))$$

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$$=_{\mathcal{M}} \;=\; \{(a,b) \mid a = b \text{ or } a,b \in \{0,1\}\}$$
- environment I: $\quad I(x) = I(y) = I(z) = 0$

Congruence axioms are essential!

$=_{\mathcal{M}}$ does not satisfy function congruence axiom $\forall x\, y.\, x = y \rightarrow f(x) = f(y)$

Remark

simplification: predicate symbols can be eliminated

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Example

formula

$$P \wedge Q(x) \wedge \neg R(x,y) \wedge x = z \to R(x,z)$$

is transformed into

$$f_P = \bullet \wedge f_Q(x) = \bullet \wedge f_R(x,y) \neq \bullet \wedge x = z \to f_R(x,z) = \bullet$$

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satisfiability in theory of equality with uninterpreted functions is undecidable

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- formula in theory of equality with uninterpreted functions

$$\bigwedge_{(\alpha,\beta)\in P} Q(\alpha(e),\beta(e)) \wedge \left(\forall v\, w. Q(v,w) \rightarrow \bigwedge_{(\alpha,\beta)\in P} Q(\alpha(v),\beta(w))\right) \rightarrow \exists z. Q(z,z)$$

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- consequence: ability of SMT solvers to prove unsat is essential

Example

- two C functions computing $x \mapsto x^3$

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int power3(int in) {
    int i, out;
    out = in;
    for (i = 0; i < 2; i++)
       out = out * in;
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SMT-LIB 2 Format for EUF

EUF formula $f(f(a)) = a \ \wedge \ f(a) = b \ \wedge \ a \neq b$

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  creates variable x of sort S

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- `declare-fun f (S1 ... Sn) T`
  creates uninterpreted function $f\colon S_1 \times \cdots \times S_n \to T$

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EUF formula $f(f(a)) = a \wedge f(a) = b \wedge a \neq b$

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(assert (= (f a) b))
```

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(assert (= (f a) b))
(assert (distinct a b))
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- `(distinct x y)` is equivalent to `not (= x y)`

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(assert (= (f (f a)) a))
(assert (= (f a) b))
(assert (distinct a b))
(check-sat)
(get-model)
```

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Outline

Congruence Closure (core algorithm for T-Solver of EUF)

input: set E of ground equations and ground equation $s \approx t$

output: valid ($E \vDash_{EUF} s = t$) or invalid ($E \nvDash_{EUF} s = t$)

Congruence Closure (core algorithm for T-Solver of EUF)

input: set E of ground equations and ground equation $s \approx t$

output: valid ($E \vDash_{EUF} s = t$) or invalid ($E \nvDash_{EUF} s = t$)

1. build congruence classes
   (a) put different subterms of terms in $E \cup \{s = t\}$ in separate sets

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   (a) put different subterms of terms in $E \cup \{s = t\}$ in separate sets
   (b) merge sets $\{\ldots, t_1, \ldots\}$ and $\{\ldots, t_2, \ldots\}$ for all $t_1 = t_2$ in E

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1. build congruence classes
   - **(a)** put different subterms of terms in $E \cup \{s = t\}$ in separate sets
   - **(b)** merge sets $\{\ldots, t_1, \ldots\}$ and $\{\ldots, t_2, \ldots\}$ for all $t_1 = t_2$ in E
   - **(c)** repeatedly merge sets

     $$\{\ldots, f(s_1, \ldots, s_n), \ldots\} \text{ and } \{\ldots, f(t_1, \ldots, t_n), \ldots\}$$

     if s_i and t_i belong to same set for all $1 \leqslant i \leqslant n$

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     $$\{\ldots, f(s_1, \ldots, s_n), \ldots\} \text{ and } \{\ldots, f(t_1, \ldots, t_n), \ldots\}$$

     if s_i and t_i belong to same set for all $1 \leqslant i \leqslant n$
2. if s and t belong to same set then return valid else return invalid

Example (1)

- set of equations E

$$f(f(f(a))) = g(f(g(f(b)))) \qquad f(g(f(b))) = f(a) \qquad g(g(b)) = g(f(a)) \qquad g(a) = b$$

equation $f(a) = g(a)$

Example (1)

- set of equations E

$$f(f(f(a))) = g(f(g(f(b)))) \qquad f(g(f(b))) = f(a) \qquad g(g(b)) = g(f(a)) \qquad g(a) = b$$

  equation $f(a) = g(a)$

- sets

1. $\{a\}$
2. $\{f(a)\}$
3. $\{b\}$
4. $\{g(b)\}$
5. $\{f(f(a))\}$
6. $\{f(f(f(a)))\}$
7. $\{f(b)\}$
8. $\{g(f(b))\}$
9. $\{f(g(f(b)))\}$
10. $\{g(f(g(f(b))))\}$
11. $\{g(g(b))\}$
12. $\{g(f(a))\}$
13. $\{g(a)\}$

Example (1)

- set of equations E

$$f(f(f(a))) = g(f(g(f(b)))) \qquad f(g(f(b))) = f(a) \qquad g(g(b)) = g(f(a)) \qquad g(a) = b$$

  equation $f(a) = g(a)$

- sets

1. $\{a\}$
2. $\{f(a)\}$
3. $\{b\}$
4. $\{g(b)\}$
5. $\{f(f(a))\}$
6. $\{f(f(f(a)))\}$
7. $\{f(b)\}$
8. $\{g(f(b))\}$
9. $\{f(g(f(b)))\}$
10. $\{g(f(g(f(b))))\}$
11. $\{g(g(b))\}$
12. $\{g(f(a))\}$
13. $\{g(a)\}$

Example (1)

- set of equations E

$$f(f(f(a))) = g(f(g(f(b)))) \qquad f(g(f(b))) = f(a) \qquad g(g(b)) = g(f(a)) \qquad g(a) = b$$

  equation $f(a) = g(a)$

- sets

1. $\{a\}$
2. $\{f(a)\}$
3. $\{b\}$
4. $\{g(b)\}$
5. $\{f(f(a))\}$
6. $\{f(f(f(a))), g(f(g(f(b))))\}$
7. $\{f(b)\}$
8. $\{g(f(b))\}$
9. $\{f(g(f(b)))\}$
11. $\{g(g(b))\}$
12. $\{g(f(a))\}$
13. $\{g(a)\}$

Example (1)

- set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))) \qquad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b}))) = \mathsf{f}(\mathsf{a}) \qquad \mathsf{g}(\mathsf{g}(\mathsf{b})) = \mathsf{g}(\mathsf{f}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{a}) = \mathsf{b}$$

  equation $\mathsf{f}(\mathsf{a}) = \mathsf{g}(\mathsf{a})$

- sets

1. $\{\mathsf{a}\}$
2. $\{\mathsf{f}(\mathsf{a})\}$
3. $\{\mathsf{b}\}$
4. $\{\mathsf{g}(\mathsf{b})\}$
5. $\{\mathsf{f}(\mathsf{f}(\mathsf{a}))\}$
6. $\{\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b}))))\}$
7. $\{\mathsf{f}(\mathsf{b})\}$
8. $\{\mathsf{g}(\mathsf{f}(\mathsf{b}))\}$
9. $\{\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))\}$
11. $\{\mathsf{g}(\mathsf{g}(\mathsf{b}))\}$
12. $\{\mathsf{g}(\mathsf{f}(\mathsf{a}))\}$
13. $\{\mathsf{g}(\mathsf{a})\}$

Example (1)

- set of equations E

$$\mathsf{f(f(f(a)))} = \mathsf{g(f(g(f(b))))} \qquad \mathsf{f(g(f(b)))} = \mathsf{f(a)} \qquad \mathsf{g(g(b))} = \mathsf{g(f(a))} \qquad \mathsf{g(a)} = \mathsf{b}$$

  equation $\mathsf{f(a)} = \mathsf{g(a)}$

- sets

1. $\{\mathsf{a}\}$
2. $\{\mathsf{f(a)}, \mathsf{f(g(f(b)))}\}$
3. $\{\mathsf{b}\}$
4. $\{\mathsf{g(b)}\}$
5. $\{\mathsf{f(f(a))}\}$
6. $\{\mathsf{f(f(f(a)))}, \mathsf{g(f(g(f(b))))}\}$
7. $\{\mathsf{f(b)}\}$
8. $\{\mathsf{g(f(b))}\}$

11. $\{\mathsf{g(g(b))}\}$
12. $\{\mathsf{g(f(a))}\}$
13. $\{\mathsf{g(a)}\}$

Example (1)

- set of equations E

$$\mathsf{f(f(f(a))) = g(f(g(f(b))))} \qquad \mathsf{f(g(f(b))) = f(a)} \qquad \mathsf{g(g(b)) = g(f(a))} \qquad \mathsf{g(a) = b}$$

  equation $\mathsf{f(a) = g(a)}$

- sets

1. $\{\mathsf{a}\}$
2. $\{\mathsf{f(a)}, \mathsf{f(g(f(b)))}\}$
3. $\{\mathsf{b}\}$
4. $\{\mathsf{g(b)}\}$
5. $\{\mathsf{f(f(a))}\}$
6. $\{\mathsf{f(f(f(a)))}, \mathsf{g(f(g(f(b))))}\}$
7. $\{\mathsf{f(b)}\}$
8. $\{\mathsf{g(f(b))}\}$

11. $\{\mathsf{g(g(b))}\}$
12. $\{\mathsf{g(f(a))}\}$

13. $\{\mathsf{g(a)}\}$

Example (1)

- set of equations E

$$f(f(f(a))) = g(f(g(f(b)))) \qquad f(g(f(b))) = f(a) \qquad g(g(b)) = g(f(a)) \qquad g(a) = b$$

  equation $f(a) = g(a)$

- sets

1. $\{a\}$
2. $\{f(a),\ f(g(f(b)))\}$
3. $\{b\}$
4. $\{g(b)\}$
5. $\{f(f(a))\}$
6. $\{f(f(f(a))),\ g(f(g(f(b))))\}$
7. $\{f(b)\}$
8. $\{g(f(b))\}$

11. $\{g(g(b)),\ g(f(a))\}$

13. $\{g(a)\}$

Example (1)

- set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))) \qquad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b}))) = \mathsf{f}(\mathsf{a}) \qquad \mathsf{g}(\mathsf{g}(\mathsf{b})) = \mathsf{g}(\mathsf{f}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{a}) = \mathsf{b}$$

  equation $\mathsf{f}(\mathsf{a}) = \mathsf{g}(\mathsf{a})$

- sets

1. $\{\mathsf{a}\}$
2. $\{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))\}$
3. $\{\mathsf{b}\}$
4. $\{\mathsf{g}(\mathsf{b})\}$
5. $\{\mathsf{f}(\mathsf{f}(\mathsf{a}))\}$
6. $\{\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b}))))\}$
7. $\{\mathsf{f}(\mathsf{b})\}$
8. $\{\mathsf{g}(\mathsf{f}(\mathsf{b}))\}$

11. $\{\mathsf{g}(\mathsf{g}(\mathsf{b})), \mathsf{g}(\mathsf{f}(\mathsf{a}))\}$

13. $\{\mathsf{g}(\mathsf{a})\}$

Example (1)

- set of equations E

$$f(f(f(a))) = g(f(g(f(b)))) \qquad f(g(f(b))) = f(a) \qquad g(g(b)) = g(f(a)) \qquad g(a) = b$$

  equation $f(a) = g(a)$

- sets

1. $\{a\}$
2. $\{f(a), f(g(f(b)))\}$
3. $\{b, g(a)\}$
4. $\{g(b)\}$
5. $\{f(f(a))\}$
6. $\{f(f(f(a))), g(f(g(f(b))))\}$
7. $\{f(b)\}$
8. $\{g(f(b))\}$

11. $\{g(g(b)), g(f(a))\}$

Example (1)

- set of equations E

$$\mathsf{f(f(f(a))) = g(f(g(f(b))))} \qquad \mathsf{f(g(f(b))) = f(a)} \qquad \mathsf{g(g(b)) = g(f(a))} \qquad \mathsf{g(a) = b}$$

  equation $\mathsf{f(a) = g(a)}$

- sets

1. $\{\mathsf{a}\}$
2. $\{\mathsf{f(a)}, \mathsf{f(g(f(b)))}\}$
3. $\{\mathsf{b}, \mathsf{g(a)}\}$
4. $\{\mathsf{g(b)}\}$
5. $\{\mathsf{f(f(a))}\}$
6. $\{\mathsf{f(f(f(a)))}, \mathsf{g(f(g(f(b))))}\}$
7. $\{\mathsf{f(b)}\}$
8. $\{\mathsf{g(f(b))}\}$

11. $\{\mathsf{g(g(b))}, \mathsf{g(f(a))}\}$

Example (1)

- set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))) \qquad \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b}))) = \mathsf{f}(\mathsf{a}) \qquad \mathsf{g}(\mathsf{g}(\mathsf{b})) = \mathsf{g}(\mathsf{f}(\mathsf{a})) \qquad \mathsf{g}(\mathsf{a}) = \mathsf{b}$$

  equation $\mathsf{f}(\mathsf{a}) = \mathsf{g}(\mathsf{a})$

- sets

1. $\{\mathsf{a}\}$
2. $\{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))\}$
3. $\{\mathsf{b}, \mathsf{g}(\mathsf{a})\}$
4. $\{\mathsf{g}(\mathsf{b})\}$
5. $\{\mathsf{f}(\mathsf{f}(\mathsf{a}))\}$
6. $\{\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{b})))), \mathsf{g}(\mathsf{g}(\mathsf{b})), \mathsf{g}(\mathsf{f}(\mathsf{a}))\}$
7. $\{\mathsf{f}(\mathsf{b})\}$
8. $\{\mathsf{g}(\mathsf{f}(\mathsf{b}))\}$

Example (1)

- set of equations E

$$\mathsf{f(f(f(a))) = g(f(g(f(b))))} \qquad \mathsf{f(g(f(b))) = f(a)} \qquad \mathsf{g(g(b)) = g(f(a))} \qquad \mathsf{g(a) = b}$$

  equation $\mathsf{f(a) = g(a)}$

- sets

  1. $\{\mathsf{a}\}$
  2. $\{\mathsf{f(a)}, \mathsf{f(g(f(b)))}\}$
  3. $\{\mathsf{b}, \mathsf{g(a)}\}$
  4. $\{\mathsf{g(b)}\}$
  5. $\{\mathsf{f(f(a))}\}$
  6. $\{\mathsf{f(f(f(a)))}, \mathsf{g(f(g(f(b))))}, \mathsf{g(g(b))}, \mathsf{g(f(a))}\}$
  7. $\{\mathsf{f(b)}\}$
  8. $\{\mathsf{g(f(b))}\}$

- conclusion: $E \nvDash_{EUF} \mathsf{f(a) = g(a)}$

Example (2)

- set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))) = \mathsf{a}$$

  equation $\mathsf{f}(\mathsf{a}) = \mathsf{a}$

Example (2)

- set of equations E

$$\mathsf{f(f(f(a)))} = \mathsf{a} \qquad \mathsf{f(f(f(f(f(a)))))} = \mathsf{a}$$

equation $\mathsf{f(a)} = \mathsf{a}$

- sets

1. $\{\mathsf{a}\}$
2. $\{\mathsf{f(a)}\}$
3. $\{\mathsf{f(f(a))}\}$
4. $\{\mathsf{f(f(f(a)))}\}$
5. $\{\mathsf{f(f(f(f(a))))}\}$
6. $\{\mathsf{f(f(f(f(f(a)))))}\}$

Example (2)

- set of equations E

$$\mathsf{f(f(f(a)))} = \mathsf{a} \qquad\qquad \mathsf{f(f(f(f(f(a)))))} = \mathsf{a}$$

  equation $\mathsf{f(a)} = \mathsf{a}$

- sets

  1. $\{\,\mathsf{a}\,\}$
  2. $\{\,\mathsf{f(a)}\,\}$
  3. $\{\,\mathsf{f(f(a))}\,\}$
  4. $\{\,\mathsf{f(f(f(a)))}\,\}$
  5. $\{\,\mathsf{f(f(f(f(a))))}\,\}$
  6. $\{\,\mathsf{f(f(f(f(f(a)))))}\,\}$

Example (2)

- set of equations E

$$f(f(f(a))) = a \qquad f(f(f(f(f(a))))) = a$$

  equation $f(a) = a$

- sets

1. $\{a,\ f(f(f(a)))\}$
2. $\{f(a)\}$
3. $\{f(f(a))\}$

5. $\{f(f(f(f(a))))\}$
6. $\{f(f(f(f(f(a)))))\}$

Example (2)

- set of equations E

$$f(f(f(a))) = a \qquad f(f(f(f(f(a))))) = a$$

  equation $f(a) = a$

- sets

  1. $\{a, f(f(f(a)))\}$
  2. $\{f(a)\}$
  3. $\{f(f(a))\}$

  5. $\{f(f(f(f(a))))\}$
  6. $\{f(f(f(f(f(a)))))\}$

Example (2)

- set of equations E

$$f(f(f(a))) = a \qquad f(f(f(f(f(a))))) = a$$

equation $f(a) = a$

- sets

1. $\{a,\ f(f(f(a))),\ f(f(f(f(f(a)))))\}$
2. $\{f(a)\}$
3. $\{f(f(a))\}$

5. $\{f(f(f(f(a))))\}$

Example (2)

- set of equations E

$$f(f(f(a))) = a \qquad f(f(f(f(f(a))))) = a$$

  equation $f(a) = a$

- sets

  1. $\{a, f(f(f(a))), f(f(f(f(f(a)))))\}$
  2. $\{f(a)\}$
  3. $\{f(f(a))\}$

  5. $\{f(f(f(f(a))))\}$

Example (2)

- set of equations E

$$f(f(f(a))) = a \qquad f(f(f(f(f(a))))) = a$$

  equation $f(a) = a$

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  1. $\{a, f(f(f(a))), f(f(f(f(f(a)))))\}$
  2. $\{f(a), f(f(f(f(a))))\}$
  3. $\{f(f(a))\}$

Example (2)

- set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))) = \mathsf{a}$$

  equation $\mathsf{f}(\mathsf{a}) = \mathsf{a}$

- sets

  1. $\{\mathsf{a}, \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a})))))\}$
  2. $\{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))\}$
  3. $\{\mathsf{f}(\mathsf{f}(\mathsf{a}))\}$

Example (2)

- set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))) = \mathsf{a}$$

  equation $\mathsf{f}(\mathsf{a}) = \mathsf{a}$

- sets

  1. $\{\mathsf{a}, \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))), \mathsf{f}(\mathsf{f}(\mathsf{a}))\}$
  2. $\{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))\}$

Example (2)

- set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))) = \mathsf{a}$$

  equation $\mathsf{f}(\mathsf{a}) = \mathsf{a}$

- sets

  1. $\{\mathsf{a}, \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))), \mathsf{f}(\mathsf{f}(\mathsf{a}))\}$
  2. $\{\mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))\}$

Example (2)

- set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))) = \mathsf{a}$$

equation $\mathsf{f}(\mathsf{a}) = \mathsf{a}$

- sets

1. $\{\mathsf{a}, \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))), \mathsf{f}(\mathsf{f}(\mathsf{a})), \mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))\}$

Example (2)

- set of equations E

$$\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))) = \mathsf{a} \qquad \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))) = \mathsf{a}$$

  equation $\mathsf{f}(\mathsf{a}) = \mathsf{a}$

- sets

  1. $\{\mathsf{a}, \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))), \mathsf{f}(\mathsf{f}(\mathsf{a})), \mathsf{f}(\mathsf{a}), \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))\}$

- conclusion: $E \vDash_{EUF} \mathsf{f}(\mathsf{a}) = \mathsf{a}$

Outline

Kröning and Strichmann

- Chapter 4
- Section 11.3

Kröning and Strichmann

- Chapter 4
- Section 11.3

Bradley and Manna

- Sections 9.1 and 9.2

Kröning and Strichmann

- Chapter 4
- Section 11.3

Bradley and Manna

- Sections 9.1 and 9.2

Important Concepts

- congruence closure
- contradictory cycle
- equality graph
- equality logic
- EUF
- uninterpreted function