universität innsbruck

SS 2024 lecture 6


# Constraint Solving

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based on a previous course by Aart Middeldorp

# Outline

## Definition (Equality Logic)

terms are restricted to variables, no quantifiers:

$$\varphi \; ::= \; \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \neg\varphi \mid t = t \qquad\qquad t \; ::= \; x$$

$T$-solver for conjunction $\varphi$ of equality logic literals over set of variables $V$

## Definition

equality graph is undirected graph $G_=(\varphi) = (V, E_=, E_{\neq})$ with

- $E_=$ edges corresponding to positive (equality) literals in $\varphi$
- $E_{\neq}$ edges corresponding to negative (inequality) literals in $\varphi$

## Lemma

$\varphi$ is satisfiable $\iff$
$G_=(\varphi)$ contains no simple contradictory cycles (with exactly one $E_{\neq}$ edge)

## Quantifier-Free Fragment of Equality Logic with Uninterpreted Functions (EUF)

$$\varphi \ ::= \ \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \neg\varphi \mid t = t \qquad\qquad t \ ::= \ f(t, \dots, t)$$

## $T$-Solver for Conjunctive Fragment via Congruence Closure

input: set $E$ of ground equations and ground equation $s \approx t$

output: valid ($E \vDash_T s = t$) or invalid ($E \nvDash_T s = t$)

1. build congruence classes
   - (a) put different subterms of terms in $E \cup \{s = t\}$ in separate sets
   - (b) merge sets $\{\dots, t_1, \dots\}$ and $\{\dots, t_2, \dots\}$ for all $t_1 = t_2$ in $E$
   - (c) repeatedly merge sets

     $$\{\dots, f(s_1, \dots, s_n), \dots\} \ \text{and} \ \{\dots, f(t_1, \dots, t_n), \dots\}$$

     if $s_i$ and $t_i$ belong to same set for all $1 \leqslant i \leqslant n$
2. if $s$ and $t$ belong to same set then return valid else return invalid

# Outline

## Difference Logic

atoms are constraints of the form

- $x - y \leqslant c$
- $x - y < c$

where $x$ and $y$ are variables and $c$ is some constant of $\mathbb{Z}$ or $\mathbb{Q}$

## Remarks

- difference logic is fragment of linear arithmetic; advantage: faster decision procedure
- domains: $\mathbb{Z}$ or $\mathbb{Q}$ ($T$-solver in polynomial time)
- $x - y = c \iff x - y \leqslant c \ \wedge\ y - x \leqslant -c$
- $x - y \geqslant c \iff y - x \leqslant -c$
- $x - y > c \iff y - x < -c$
- $x < c \iff x - x_0 < c$ where $x_0$ is fresh variable that must be assigned 0

## Example Job-Shop Scheduling

- $m$ machines $(M_1, \ldots, M_m)$ and $n$ jobs $(J_1, \ldots, J_n)$
- each job $J_i$ is sequence $(M^i_1, d^i_1), \ldots, (M^i_{n_i}, d^i_{n_i})$ of operations consisting of machine and duration (rational number; $\tau(M, d) = d$)
- $O$ is multiset of all operations from all jobs
- schedule is function $S$ that defines for each operation $v \in O$ its starting time $S(v)$ on machine specified by $v$
- schedule $S$ is feasible if

$$S(v) \geqslant 0 \qquad \text{for all } v \in O$$

$$S(v_i) + \tau(v_i) \leqslant S(v_j) \qquad \text{for all consecutive } v_i, v_j \text{ in same job}$$

$$S(v_i) + \tau(v_i) \leqslant S(v_j) \ \vee \ S(v_j) + \tau(v_j) \leqslant S(v_i)$$

for every pair of different operations $v_i, v_j$ scheduled on same machine

- aim: minimize global duration $gd$; add constraints $S(v) + \tau(v) \leq gd$ for each $v \in O$

## Definition Inequality Graph

conjunction $\varphi$ of nonstrict difference constraints

- inequality graph of $\varphi$ contains edge from $x$ to $y$ with weight $c$ for every constraint $x - y \leqslant c$ in $\varphi$

## Theorem

conjunction $\varphi$ of nonstrict difference constraints is satisfiable $\iff$ inequality graph of $\varphi$ has no negative cycle

## Example

$$\begin{aligned} x - y &\leqslant 2 \\ y - z &\leqslant -3 \\ z - x &\leqslant 2 \end{aligned}$$

$$\begin{tikzcd} x \arrow[r, "2"] & y \arrow[r, "-3"] & z \arrow[ll, bend left, "2"] \end{tikzcd}$$

satisfiable

## Theorem

conjunction $\varphi$ of nonstrict difference constraints is satisfiable $\iff$
inequality graph of $\varphi$ has no negative cycle

## Proof

$\Rightarrow$ negative cycle

$$x_1 \xrightarrow{k_1} x_2 \xrightarrow{k_2} x_3 \longrightarrow \cdots \longrightarrow x_n \xrightarrow{k_n} x_1$$

in inequality graph of $\varphi$ corresponds to conjuction

$$x_1 - x_2 \leqslant k_1 \ \wedge \ x_2 - x_3 \leqslant k_2 \ \wedge \ \cdots \ \wedge \ x_n - x_1 \leqslant k_n$$

adding these literals gives

$$0 \leqslant k_1 + k_2 + \cdots + k_n$$

with $k_1 + k_2 + \cdots + k_n < 0$ $\lightning$

## Theorem

conjunction $\varphi$ of nonstrict difference constraints is satisfiable $\iff$
inequality graph of $\varphi$ has no negative cycle

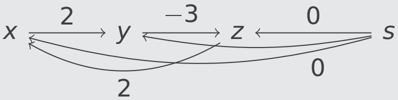
## Proof

$\Longleftarrow$ assume inequality graph of $\varphi$ has no negative cycle

construct satisfying assignment for $\varphi$ as follows

- add additional starting node $s$ in graph, add edges $s \to x$ with weight 0 for all variables $x$
- define $v(x) = -\text{distance}(s, x)$; well-defined, since there are no negative cycles
- $v$ satisfies $\varphi$

## Example




$v(x) = 1, v(y) = 0, v(z) = 3$

## Algorithms for Distance Computation and Negative Cycle Detection

- Dijkstra
  - computes distances from a single source to all other nodes
  - complexity: $\mathcal{O}(|V| \cdot log(|V|) + |E|)$
  - restriction: no negative cycles allowed
- Bellman–Ford
  - computes distances from a single source to all other nodes
  - complexity: $\mathcal{O}(|V| \cdot |E|)$
  - can also detect negative cycles
- Floyd–Warshall
  - computes distances between all nodes
  - complexity: $\mathcal{O}(|V|^3)$
  - can also detect negative cycles

$\Rightarrow$ use Bellman–Ford algorithm for difference logic

# Bellman–Ford Algorithm for Inequality Graphs

**Input** inequality graph $(V, E, w)$ with fresh starting node $s$

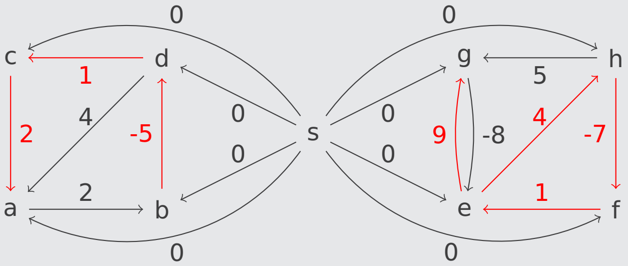
**Output** "$\exists$ negative cycle" or distances to node $s$

1. distance$[v] := 0$ for all nodes $v \in V$ (this step is special for inequality graphs)

2. repeat $|V| - 1$ times
   for all $(u, v) \in E$ do
     if distance$[v] >$ distance$[u] + w(u, v)$ then
       distance$[v] :=$ distance$[u] + w(u, v)$
       predecessor$[v] := u$

3. for all $(u, v) \in E$ do
   if distance$[v] >$ distance$[u] + w(u, v)$ then
     return "$\exists$ negative cycle"
         which can be reconstructed using predecessor array

4. return distance array, shortest paths available via predecessor array




| iteration | a | b | c | d | e | f | g | h |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | -5 | -8 | -7 | 0 | 0 |
| 2 | -1 | 0 | -4 | -5 | -8 | -7 | 0 | -4 |
| 3 | -2 | 0 | -4 | -5 | -8 | -11 | 0 | -4 |
| 4 | -2 | 0 | -4 | -5 | -10 | -11 | 0 | -4 |
| 5 | -2 | 0 | -4 | -5 | -10 | -11 | -1 | -6 |

. . . 2 more iterations, then negative cycle is detected

# Outline

## Definition (Theory of Linear Arithmetic over $D$)

- for variables $x_1, \ldots, x_n$, built quantifier-free formulas according to grammar

$$\varphi ::= \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid t < t \mid t \leq t \qquad s = t \text{ is encoded as } s \leq t \wedge t \leq s$$

$$t ::= a_1x_1 + \cdots + a_nx_n + b \qquad \text{for } a_1, \ldots, a_n, b \in \text{ in domain } D$$

- **solution** assigns values in $D$ to $x_1, \ldots, x_n$

## Definitions

- **Linear Real Arithmetic** (**LRA**) uses domain $D = \mathbb{R}$
- **Linear Integer Arithmetic** (**LIA**) uses domain $D = \mathbb{Z}$

## Example

- $x + y + z = 2 \ \wedge\ z > y \ \wedge\ y > -1$
  is satisfiable in LRA and LIA, e.g. with $v(x) = v(y) = 0$ and $v(z) = 2$
- $x < 3 \ \wedge\ 2x > 4$
  is unsatisfiable in LIA but satisfiable in LRA, e.g. with $v(x) = 2.5$

## Relevance of Linear Arithmetic

LRA and LIA admit more natural and succinct encodings of

- everything with cardinality constraints: $n$-queens, Sudoku, Minesweeper, ...
- planning problems
- scheduling problems
- component configuration problems
- ...

## $\mathcal{T}$-Solver for Conjunctions of Linear Arithmetic Inequalities

- integers (LIA): **NP-complete**
- reals or rationals (LRA): **polynomial** **Simplex algorithm**

**exponential worst-case complexity**

## Some History

**1947** **Danzig** proposed Simplex algorithm to solve optimization problem:

$$\text{maximize } c(\vec{x}) \qquad \text{such that} \qquad A\vec{x} \leq b \text{ and } \vec{x} \geq 0$$

for linear objective function $c$, matrix $A$, vector $b$, and vector of variables $\vec{x}$

- also known as linear programming

**1979** **Khachiyan** proposed polynomial version based on ellipsoid method

**1984** **Karmakar** proposed polynomial version based on interior points method

**2000-** $\mathcal{T}$-solver version of simplex algorithm to solve satisfiability problem

$$A\vec{x} \leq b$$

## Problem Input (General Form)

- $m$ equalities $\qquad a_1x_1 + \cdots + a_nx_n = 0$
- (optional) lower and upper bounds on variables $\qquad l_i \leq x_i \leq u_i$

## Lemma

*any conjunctive LRA problem without $<$ can be turned into equisatisfiable general form*

## Example

$$\begin{aligned} x - y &\geq -1 \\ y &\leq 4 \\ x + y &\geq 6 \\ 3x - y &\leq 7 \end{aligned} \quad\Longrightarrow\quad \begin{aligned} -x + y - s_1 &= 0 & s_1 &\leq 1 \\ y - s_2 &= 0 & s_2 &\leq 4 \\ -x - y - s_3 &= 0 & s_3 &\leq -6 \\ 3x - y - s_4 &= 0 & s_4 &\leq 7 \end{aligned}$$

slack variables

- $s_1, s_2, s_3, s_4$ are slack variables
- $x, y$ are problem variables

## Representation

- represent equalities

$$\begin{aligned} -x + y - s_1 &= 0 \\ y - s_2 &= 0 \\ -x - y - s_3 &= 0 \\ 3x - y - s_4 &= 0 \end{aligned}$$

via $m \times n$ matrix presentation

$$\begin{array}{c@{\quad}c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} & \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

$\leftarrow$ **nonbasic** variables

**basic** variables $\rightarrow$

## Notation

- matrix is **tableau**
- $B$ is set of **basic variables** (in tableau listed vertically)
- $N$ is set of **nonbasic variables** (in tableau listed horizontally)

# Simplex Algorithm as $T$-solver

**Input:** conjunction of LRA atoms $\varphi$ without $<$
**Output:** satisfiable assignment or unsatisfiable

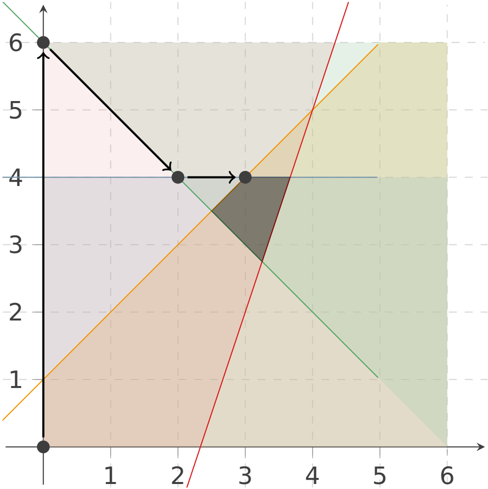
1. transform $\varphi$ into general form and construct tableau
2. fix order on variables and assign 0 to each variable
3. if all basic variables satisfy their bounds then return **current (satisfying) assignment**
4. let $x_i \in B$ be variable that violates its bounds
5. search for suitable variable $x_j \in N$ for pivoting with $x_i$
6. return **unsatisfiable** if search unsuccessful
7. perform **pivot** operation on $x_i$ and $x_j$
9. update assignment
10. go to step 3

# Simplex, Visually

- constraints

$$x - y \geq -1$$
$$y \leq 4$$
$$x + y \geq 6$$
$$3x - y \leq 7$$

- solution space
- Simplex solution search



## Example

| tableau | bounds | assignment |
|---|---|---|

$$\begin{array}{c} \\ s_3 \\ x \\ y \\ s_4 \end{array} \begin{array}{c} \begin{array}{cc} s_2 & s_1 \end{array} \\ \begin{pmatrix} -2 & 1 \\ 1 & -1 \\ 1 & 0 \\ 2 & -3 \end{pmatrix} \end{array} \qquad \begin{array}{l} s_1 \leqslant 1 \\ s_2 \leqslant 4 \\ s_3 \leqslant -6 \\ s_4 \leqslant 7 \end{array}$$

| $x$ | $y$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|---|
| 3 | 4 | 1 | 4 | −7 | 5 |

**3** Iteration 3

- $s_1$ violates its bounds
- decreasing $s_1$ requires to decrease $s_2$ or $s_3$ (both suitable since they have no lower bound)
- pivot $s_1$ with $s_3$:

$$s_3 = s_1 - 2s_2 \qquad x = -s_1 + s_2$$
$$y = s_2 \qquad s_4 = -3s_1 + 2s_2$$

- update assignment

$$s_1 := 1 \qquad s_2 = 4$$
$$s_3 := -7 \qquad x := 3 \qquad y := 4 \qquad s_4 := 5$$

# Simplex Algorithm as $T$-solver

$$A\vec{x}_N = \vec{x}_B \qquad (1)$$

$$-\infty \le l_i \le x_i \le u_i \le +\infty \qquad (2)$$

nonbasic $\vec{x}_N$

basic $\vec{x}_B$

$$\begin{array}{cc} & \begin{array}{ccc} \phantom{\cdots} & x_j & \phantom{\cdots} \end{array} \\ x_i & \begin{pmatrix} \cdots & & \cdots \\ & A_{ij} & \\ \cdots & & \cdots \end{pmatrix} \end{array}$$

## Invariant

- (1) is satisfied and (2) holds for all nonbasic variables

## Pivoting

- swap basic $x_i$ and nonbasic $x_j$, so $i \in B$ and $j \in N$

$$x_i = \sum_{k \in N} A_{ik} x_k \qquad \Longrightarrow \qquad x_j = \frac{1}{A_{ij}} \left( x_i - \sum_{k \in N - \{j\}} A_{ik} x_k \right) \qquad (\star)$$

- new tableau $A'$ consists of $(\star)$ and $A_{B-\{i\}}\vec{x}_N = \vec{x}_{B-\{i\}}$ with $(\star)$ substituted

## Update

- assignment of $x_i$ is updated to previously violated bound $l_i$ or $u_i$,
- assignment of $x_k$ is recomputed using $A'$ for all $k \in B - \{i\} \cup \{j\}$

## Simplex Algorithm as $T$-solver

$$A\vec{x}_N = \vec{x}_B \tag{1}$$

$$\forall i \in N.\ -\infty \leq l_i \leq x_i \leq u_i \leq +\infty \tag{2}$$

### Suitability

- basic variable $x_i$ violates lower or upper bound
- pick nonbasic variable $x_j$ such that
  - if $u_i < l_i$: problem is trivially unsatisfiable and no suitable $x_j$ exists
  - if $x_i < l_i$: $A_{ij} > 0$ and $x_j < u_j$ or $A_{ij} < 0$ and $x_j > l_j$
  - if $x_i > u_i$: $A_{ij} > 0$ and $x_j > l_j$ or $A_{ij} < 0$ and $x_j < u_j$

### Observation

- problem is unsatisfiable if no suitable pivot exists

# Outline

1. Summary of Previous Lecture
2. Difference Logic
3. Simplex Algorithm
4. **Support of Strict Inequalities**
5. Further Reading

## Motivation

strict inequalities naturally arise, e.g., as negated non-strict inequalities in DPLL($T$)

$$\neg(x + 3 \leq 5y) \quad \leftrightarrow \quad x + 3 > 5y$$

## How to Treat Strict Inequalities

replace in conjunction of inequalities $C$ every strict inequality

$$a_1x_1 + \cdots + a_nx_n > b \qquad\qquad a_1x_1 + \cdots + a_nx_n < b$$

by non-strict inequality

$$a_1x_1 + \cdots + a_nx_n \geq b + \delta \qquad\qquad a_1x_1 + \cdots + a_nx_n \leq b - \delta$$

to obtain constraints $C_\delta$ in LRA without $>$ and $<$, and treat $\delta$ symbolically during simplex algorithm ($\delta$ represents small positive rational number)

## Lemma

*$C$ is satisfiable $\iff$ $\exists$ rational number $\delta > 0$ such that $C_\delta$ is satisfiable*

## Symbolical computation with $\delta$

- $\delta$ represents small positive rational number, i.e., smaller than every concrete rational number that occurs during the computations of the simplex algorithm
- treat $\delta$ symbolically: $\mathbb{Q}_\delta = \{(c,k) \mid c,k \in \mathbb{Q}\}$ with $(c,k)$ representing $c + k\delta$
- operations for all $a, c_1, k_1, c_2, k_2 \in \mathbb{Q}$
  - addition: $(c_1,k_1) + (c_2,k_2) = (c_1 + c_2, k_1 + k_2)$
  - multiplication: $a \cdot (c_1,k_1) = (ac_1, ak_1)$
  - equality: $(c_1,k_1) = (c_2,k_2) \quad \leftrightarrow \quad c_1 = c_2 \wedge k_1 = k_2$
  - comparison: $(c_1,k_1) < (c_2,k_2) \quad \leftrightarrow \quad c_1 < c_2 \vee c_1 = c_2 \wedge k_1 < k_2$
    $(c_1,k_1) \leq (c_2,k_2) \quad \leftrightarrow \quad c_1 < c_2 \vee c_1 = c_2 \wedge k_1 \leq k_2$
- multiplication of two $\mathbb{Q}_\delta$-numbers is not defined, but also not required for the simplex algorithm
  - coefficients in the tableau stay in $\mathbb{Q}$
  - only bounds and assignment require $\mathbb{Q}_\delta$

## Example

tableau

$$\begin{array}{c} \\ x \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & y \end{array} \\ \begin{pmatrix} 1 & -1 \\ 2 & -3 \\ -1 & 3 \end{pmatrix} \end{array}$$

constraints

$$\begin{aligned} (2,1) &< s_1 \\ (0,0) &\leqslant s_2 \\ (1,0) &\leqslant s_3 \end{aligned}$$

assignment

| $x$ | $y$ | $s_1$ | $s_2$ | $s_3$ |
|---|---|---|---|---|
| $(2,1)$ | $(0,0)$ | $(2,1)$ | $(4,2)$ | $(-2,-1)$ |

- pivot $s_1$ with $x$ $\Longrightarrow$

$$\begin{aligned} x &= s_1 - y \\ s_2 &= 2(s_1 - y) - y \quad = 2s_1 - 3y \\ s_3 &= -(s_1 - y) + 2y = -s_1 + 3y \end{aligned}$$

# Outline

## Kröning and Strichmann

- Sections 5.1, 5.2 and 5.7

## Further Reading


Bruno Dutertre and Leonardo de Moura.
A Fast Linear-Arithmetic Solver for DPLL(T)
In Proc. of International Conference on Computer Aided Verification, pp. 81–94, 2006.


## Important Concepts

- basic and nonbasic variables
- Bellman–Ford algorithm
- difference logic
- linear arithmetic (LRA and LIA)
- negative cycles
- pivoting
- $\mathbb{Q}_\delta$
- simplex algorithm
- suitable pair of variables
- tableau