universität innsbruck

SS 2024 lecture 7

Constraint Solving

René Thiemann and Fabian Mitterwallner

based on a previous course by Aart Middeldorp

Outline

1. **Summary of Previous Lecture**
2. **Complexity of Simplex Algorithm**
3. **Unsatisfiable Cores and Farkas' Lemma**
4. **Simplex Algorithm for DPLL($\mathcal{T}$)**
5. **Further Reading**

Difference Logic

conjunction of constraints of the form $x - y \leqslant c$ or $x - y < c$

Definition Inequality Graph

conjunction φ of nonstrict difference constraints

- inequality graph of φ contains edge from $x \xrightarrow{c} y$ for every constraint $x - y \leqslant c$ in φ

Theorem

conjunction φ of nonstrict difference constraints is satisfiable $\iff$
inequality graph of φ has no negative cycle

Bellman-Ford Algorithm

computes distances in graphs from single source; detects negative cycles

Simplex – Representation

- represent m inequalities using m slack variables s_i and bounds $s_i \leq / \geq c$

$$\begin{aligned} -x + y &\leq 1 \\ y &\leq 4 \\ -x - y &\leq -6 \\ 3x - y &\leq 7 \end{aligned} \quad \Longrightarrow \quad \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \quad \begin{aligned} s_1 &\leq 1 \\ s_2 &\leq 4 \\ s_3 &\leq -6 \\ s_4 &\leq 7 \end{aligned}$$

- matrix presentation

$$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} & \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 3 & -1 \end{pmatrix} \end{array}$$

$\leftarrow$ **nonbasic** variables

basic variables $\rightarrow$

meaning of rows:

equalities, e.g.,

$s_4 = 3x - 1y$

Notation

- matrix is **tableau**, stored in combination with **bounds** $x \leq / \geq c$ and **assignment**
- B is set of **basic variables** (in tableau listed vertically)
- N is set of **nonbasic variables** (in tableau listed horizontally)

DPLL(T) Simplex Algorithm

Input: conjunction of LRA atoms φ without $<$
Output: satisfiable assignment or unsatisfiable

1. transform φ into tableau and bounds
2. assign 0 to each variable
3. if all basic variables satisfy their bounds then return current (satisfying) assignment
4. let $x_i \in B$ be variable that violates its bounds
5. search for suitable variable $x_j \in N$ for pivoting with x_i
6. return unsatisfiable if search unsuccessful
7. perform pivot operation on x_i and x_j
9. update assignment
10. go to step 3

DPLL(T) Simplex Algorithm

$$A\vec{x}_N = \vec{x}_B \qquad (1)$$

$$-\infty \leq l_i \leq x_i \leq u_i \leq +\infty \qquad (2)$$

$$\begin{array}{cc} & \text{nonbasic } \vec{x}_N \\ & x_j \\ \text{basic } \vec{x}_B \quad x_i & \begin{pmatrix} \dots & & \dots \\ & A_{ij} & \\ \dots & & \dots \end{pmatrix} \end{array}$$

Invariant

- (1) is satisfied and (2) holds for all nonbasic variables

Suitability

- for $x_i \in B$ violating lower or upper bound, find suitable non-basic variable x_j such that increase (or decrease) of x_j is possible w.r.t. bounds of x_j and helps to solve violation of x_i

Pivoting

- swap basic x_i and nonbasic x_j, so $i \in B$ and $j \in N$
- reorder row i in tableau to obtain form $x_j = \ldots$ ($\star$), and substitute ($\star$) in remaining tableau
- result afterwards: tableau A' where $j \in B$ and $i \in N$

Update

- assignment of x_i is updated to previously violated bound l_i or u_i,
- assignment of each $x_k \in B$ is recomputed using A'

$\mathbb{Q}_\delta$: δ-Rationals

- δ-rationals are used for supporting strict inequalities in LRA and difference logic

$$\text{replace} \quad \mathit{expr} < c \quad \text{by} \quad \mathit{expr} \leq c - \delta$$

- δ represents some small positive rational number
- computation on $\mathbb{Q}_\delta$ is done symbolically, e.g., in simplex algorithm
- after solution for $\mathbb{Q}_\delta$ is detected, a concrete δ can be computed (exercise)

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Bland's Rule

- in pivoting pick lexicographically smallest $(x_i, x_j) \in B \times N$ such that x_i and x_j are suitable; assumes some fixed order on variables

Example (Felgenhauer and Middeldorp)

$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$

$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} x_3 \\ x_4 \end{array} & \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array}$$

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$$-1 \leqslant x_1 \leqslant 0 \qquad -4 \leqslant x_2 \leqslant 0 \qquad -5 \leqslant x_3 \leqslant -4 \qquad -7 \leqslant x_4 \leqslant 1$$

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$$\begin{array}{cc} & \begin{matrix} x_1 & x_4 \end{matrix} \\ \begin{matrix} x_3 \\ x_2 \end{matrix} & \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \end{array} \quad \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \hline 0 & 1 & 2 & 1 \end{array}$$

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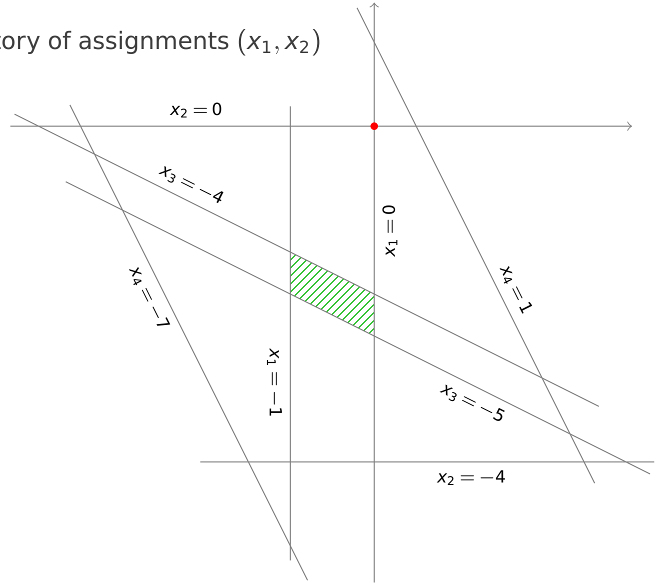
trajectory of assignments (x_1, x_2)



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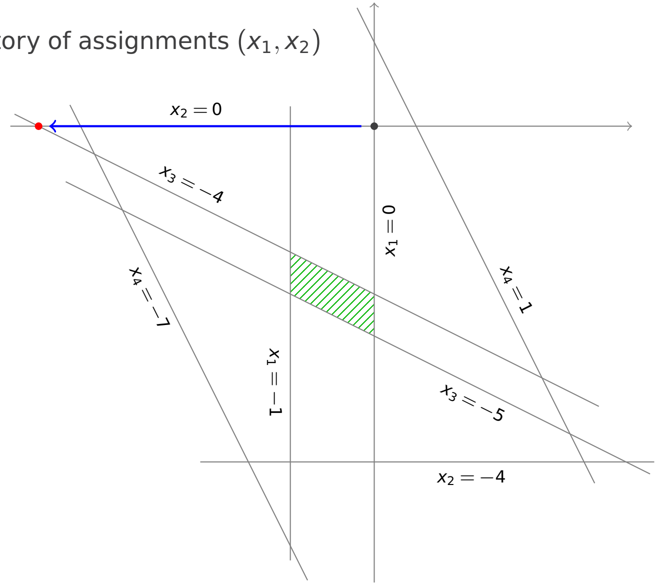

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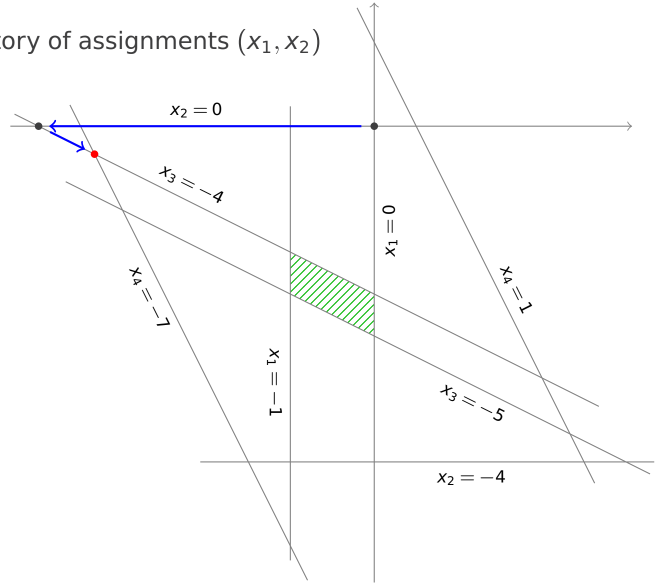

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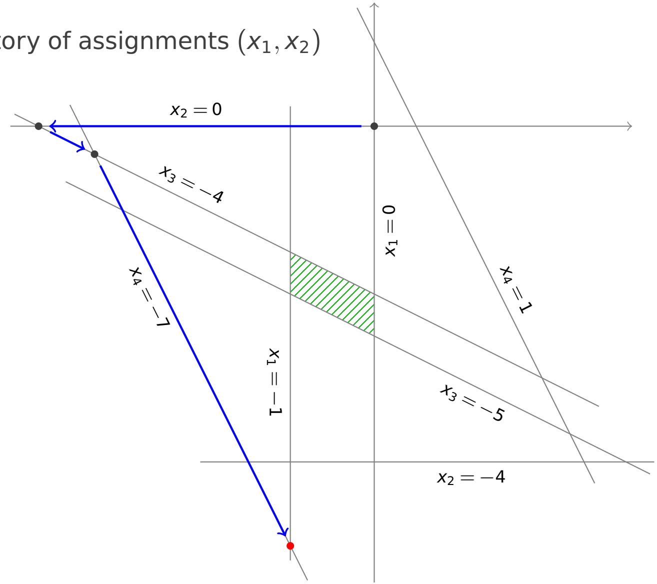

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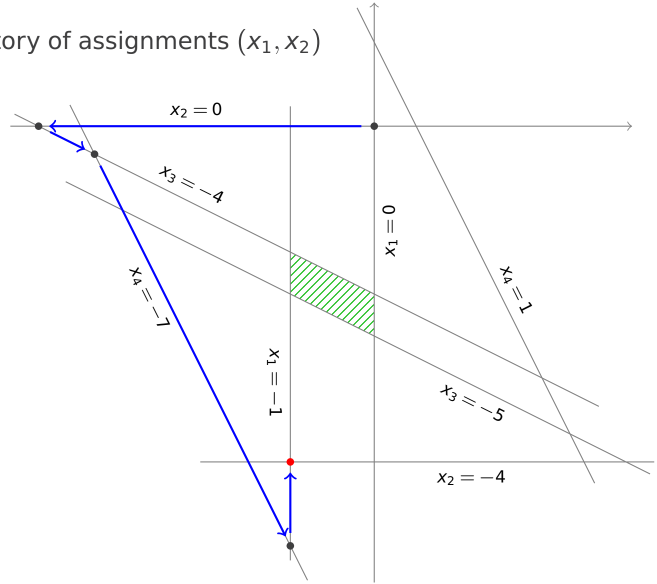

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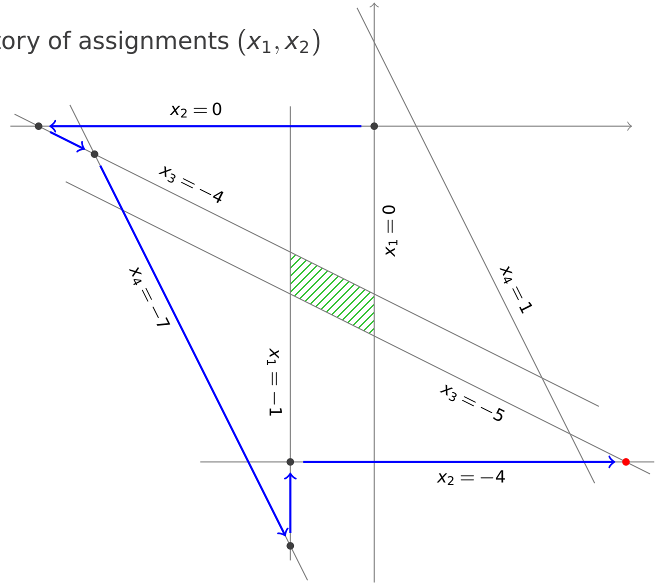

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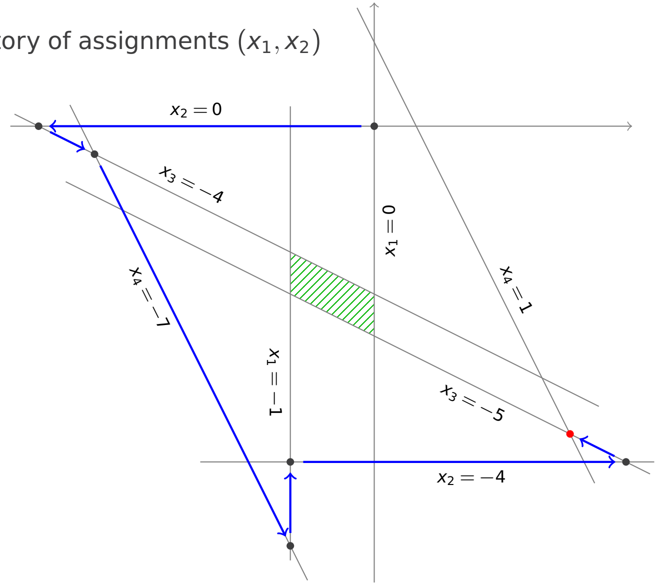

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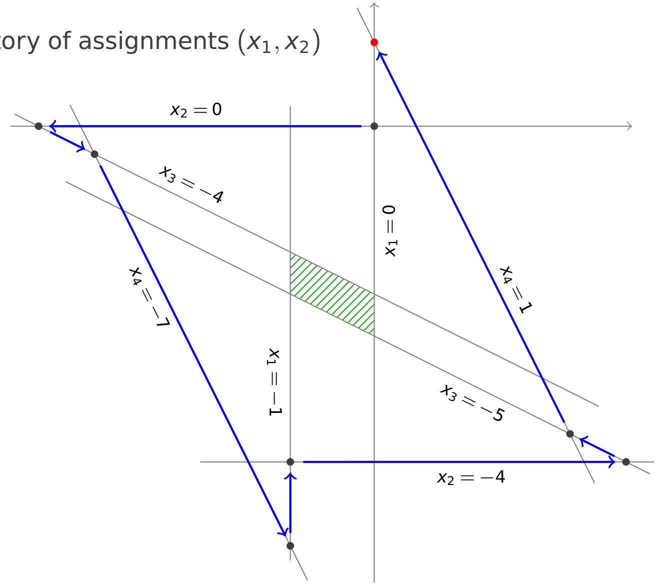

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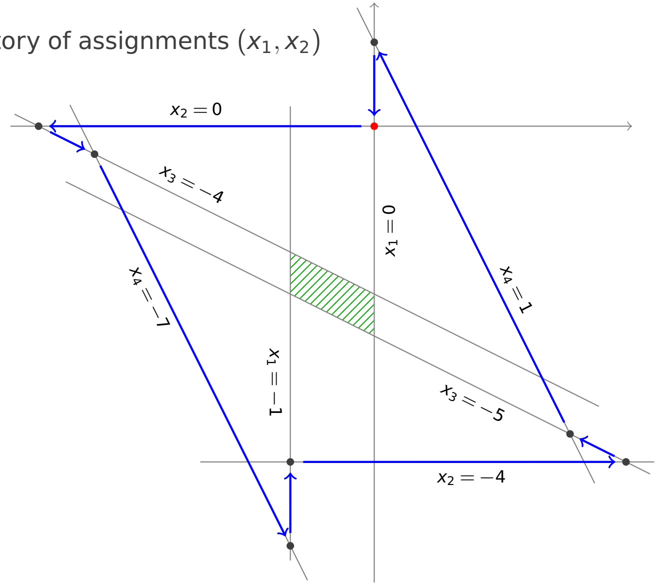

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$$\downarrow$$

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violation of Bland's rule

$$\downarrow$$

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satisfying assignment

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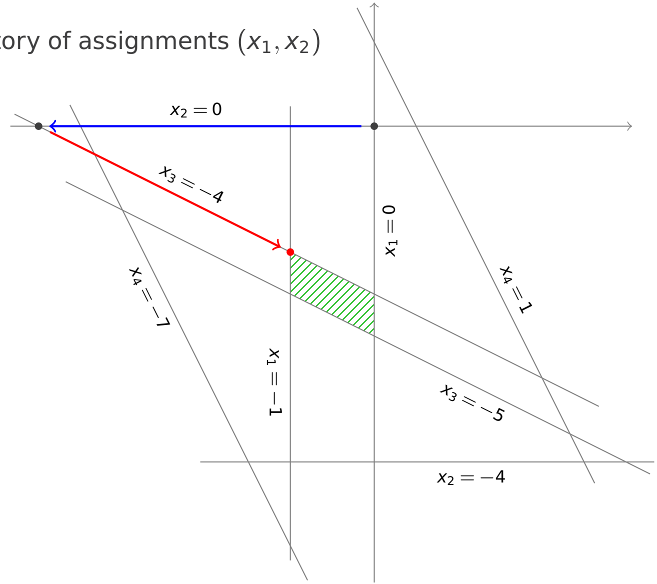

Outline

Unsatisfiable Cores for DPLL(T)

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    is an unsatisfiable core
  - this core is minimal w.r.t. the subset-relation

Farkas' Lemma

- consider $\leq$-constraints $\underbrace{-x \leq -5}_{\ell_1 \leq r_1} \wedge \underbrace{2x + y \leq 12}_{\ell_2 \leq r_2} \wedge \underbrace{-y \leq -3}_{\ell_3 \leq r_3} \wedge \underbrace{x - 3y \leq 2}_{\ell_4 \leq r_4}$
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  - Farkas' Lemma: finite set of $\leq$-constraints is unsatisfiable iff Farkas' coefficients exists
    - soundness: existence of Farkas' coefficients obviously shows unsatisfiability
    - completeness: if constraints are unsatisfiable, then simplex will detect this; whenever unsatisfiability is detected in simplex algorithm, one can extract Farkas' coefficients from the tableau equation (similar to the detection of unsatisfiable cores, but with finer analysis)

Example (Application of Linear Arithmetic: Termination Proving)

- consider program

```
factorial(n) {
  i = 1;
  r = 1;
  while (i <= n) {
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- proving termination: find linear ranking function, i.e., linear expression $e(i, n, r)$, decrease factor d with $0 < d \in \mathbb{Q}$, and bound $f \in \mathbb{Q}$ such that
  - $\varphi \to e(i, n, r) \geq e(i', n', r') + d$ (expression decreases in every iteration by at least d)
  - $\varphi \to e(i, n, r) \geq f$ (expression is bounded from below by f)

Example (Termination Proof Continued)

- loop iteration $\varphi := i \leq n \wedge i' = i + 1 \wedge r' = r \cdot i \wedge n' = n$
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  - $i < n \wedge i' = i + 1 \wedge n' = n \wedge n - i < n' - i' + 1$ (violate decrease)
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Example (Termination Proof Continued)

- loop iteration $\varphi := i \leq n \wedge i' = i + 1 \wedge r' = r \cdot i \wedge n' = n$
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- choose linear ranking function $e(i,n,r) := n - i$ and $d := 1$ and $f = -1$, and drop all non-linear constraints to get two linear problems:
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- solution: combined search for $e(i,n,r)$, d and f and Farkas' coefficients

Towards Algorithm to Synthesize Linear Ranking Function

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$$\underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}}_{A} \cdot \begin{pmatrix} i \\ n \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}}_{A'} \cdot \begin{pmatrix} i' \\ n' \end{pmatrix} \leq \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\vec{b}}$$

encodes transition formula $i \leq n \wedge n' = n \wedge i' = i + 1$
(formula φ after removal of non-linear part)

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- soundness: extract parameters of ranking function from concrete solution $\vec{c_1}, \vec{c_2}$

$$e(\vec{x}) = \vec{c_2}A'\vec{x} \qquad d = -\vec{c_2}\vec{b} \qquad f = -\vec{c_1}\vec{b}$$

Example Application (Continue in Termination Proof)

- $\begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix} \geq \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} c_6 & c_7 & c_8 & c_9 & c_{10} \end{pmatrix} \geq \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}$
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- extract ranking function parameters
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  - $f = -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} \vec{b} = 0$

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- loop transition formula: $A\vec{x} + A'\vec{x'} \leq \vec{b}$
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  - hence $e(\vec{x}) \geq f$
- decrease $e(\vec{x}) \geq e(\vec{x'}) + d$: similar to boundedness

Outline

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  - simplex can be used as backend for LIA solver (next lecture), where new constraints are added on-the-fly
  - aim: do not restart simplex from scratch when slightly modifying constraints

Incremental Interface for DPLL(T)

- make all constraints available to simplex at the beginning of the algorithm, ignoring Boolean structure $\Longrightarrow$ obtain global tableau for all constraints

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  - usually occurs after a conflict has been detected
  - tableau and assignment can be reused from before: they satisfy invariants (1) and (2)

Example (Simplex as DPLL(T) Theory Solver)

- input is formula φ

$$(c_1 \vee c_5) \wedge (c_6 \vee \neg c_3) \wedge (c_4 \vee c_5) \wedge (c_2 \vee c_7 \vee \neg c_8) \wedge \dots$$

$$\underbrace{x \geq 5}_{c_1} \qquad \underbrace{2x + y \leq 12}_{c_2} \qquad \underbrace{y < 3}_{c_3} \qquad \underbrace{x - 3y \leq 2}_{c_4} \qquad \dots$$

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  - start simplex with tableau for all atoms and negated atoms within φ, but no activated bounds (obtain tableau T_0, assignment $v_0(x) = 0$)

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  - extend to $c_1, \neg c_3, c_4, c_2$, activate bound, execute simplex on T_1 and v_1 to obtain T_2 and v_2

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  - ... next simplex invocation starting from T_2 and v_2 ...

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- `SimplexIOInterface.hs` – wrapper for incremental simplex algorithm + example

`ghci SimplexIOInterface.hs`

```
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```

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```

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ghci>
```

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Okay <state 0, asserted: []>
ghci> assert 1
Okay <state 1, asserted: [1]>
ghci>
```

```
ghci SimplexIOInterface.hs
ghci> initSimplex exampleSlidesInput
Okay <state 0, asserted: []>
ghci> assert 1
Okay <state 1, asserted: [1]>
ghci> assert (-3)
Okay <state 2, asserted: [-3,1]>
ghci> assert 4
Okay <state 3, asserted: [4,-3,1]>
ghci>
```

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Okay <state 0, asserted: []>
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Okay <state 2, asserted: [-3,1]>
ghci> assert 4
Okay <state 3, asserted: [4,-3,1]>
ghci> check
Okay <state 4, asserted: [4,-3,1]>
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ghci> assert 4
Okay <state 3, asserted: [4,-3,1]>
ghci> check
Okay <state 4, asserted: [4,-3,1]>
ghci> checkpoint
checkpoint "cp5" created
Okay <state 5, asserted: [4,-3,1]>
ghci>
```

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Okay <state 3, asserted: [4,-3,1]>
ghci> check
Okay <state 4, asserted: [4,-3,1]>
ghci> checkpoint
checkpoint "cp5" created
Okay <state 5, asserted: [4,-3,1]>
ghci> assert 2
Okay <state 6, asserted: [2,4,-3,1]>
ghci> check
unsat-core [2,1,-3] detected, use backtrack to one of ["cp5"]
ghci>
```

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ghci> assert 2
Okay <state 6, asserted: [2,4,-3,1]>
ghci> check
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ghci> backtrack "cp5"
Okay <state 7, asserted: [4,-3,1]>
```

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- linear programming: find solution in $\mathbb{Q}$ of linear constraints φ that maximizes linear function f

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[a] Alexander Schrijver, Theory of Linear and Integer Programming, Chapter 11

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- solution: use standard simplex algorithm for linear programming,[a] and not the DPLL(T)-variant of simplex for decidability that was presented here
  - at least one difference: solving $A\vec{x} \leq \vec{b}$ is formulated via slack variables $\vec{s} \geq \vec{0}$ as tableau $A\vec{x} + \vec{s} = \vec{b}$ so the tableau equations can have non-zero constants

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Outline

1. **Summary of Previous Lecture**
2. **Complexity of Simplex Algorithm**
3. **Unsatisfiable Cores and Farkas' Lemma**
4. **Simplex Algorithm for DPLL($\mathcal{T}$)**
5. **Further Reading**

Kröning and Strichmann

- Sections 5.1 and 5.2

Further Reading

Bruno Dutertre and Leonardo de Moura.
A Fast Linear-Arithmetic Solver for DPLL(T)
In Proc. CAV, LNCS 4144, pp. 81–94, 2006.

Bertram Felgenhauer and Aart Middeldorp
Constructing Cycles in the Simplex Method for DPLL(T)
Proc. 14th ICTAC, LNCS 10580, pp. 213–228, 2017

Andreas Podelski and Andrey Rybalchenko
A Complete Method for the Synthesis of Linear Ranking Functions
Proc. VMCAI 2004, LNCS 2937, pp. 239–251, 2004

Ralph Bottesch, Max W. Haslbeck, and René Thiemann
Verifying an Incremental Theory Solver for Linear Arithmetic in Isabelle/HOL
In Proc. FroCoS, LNAI 11715, pp. 223—239, 2019.

Important Concepts

- active and inactive bounds
- Bland’s selection rule
- Farkas’ coefficients
- Farkas’ lemma
- incremental simplex algorithm
- linear programming
- linear ranking function
- unsatisfiable core