universität innsbruck

SS 2024 lecture 8

Constraint Solving

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based on a previous course by Aart Middeldorp

Outline

Properties of DPLL(T) Simplex Algorithm

- termination ensured via Bland's rule:
  choose x_i and x_j for pivoting in a way that $(x_i, x_j) \in B \times N$ is lexicographically smallest

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- ranking functions for proving termination can be synthesized
- DPLL(T) simplex not well suited for linear programming, i.e., optimization problems

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Example (Application of Linear Arithmetic: Termination Proving)

- last lecture

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int factorial(int n) {
  int i = 1;
  int r = 1;
  while (i <= n) {
    r = r * i;
    i = i + 1;   }
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- remark: ranking function formula consists purely of $\leq$ inequalities
  - $\varphi := i \leq n \wedge n' = n \wedge i' = i + 1$
  - $\varphi \to e(i, n) \geq e(i', n') + d$
  - $\varphi \to e(i, n) \geq f$

Example (Application of Linear Integer Arithmetic: Termination Proving)

- consider another program

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int log2(int x)      {
    int n := 0;
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- remark: LIA reasoning is crucial, the problem is not wrong choice of expression e; program does not terminate when executed with real number arithmetic

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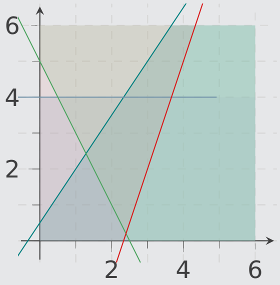
Example

$$3x - 2y \geq -1$$
$$y \leq 4$$
$$2x + y \geq 5$$
$$3x - y \leq 7$$

- looking for solution in $\mathbb{Z}^2$



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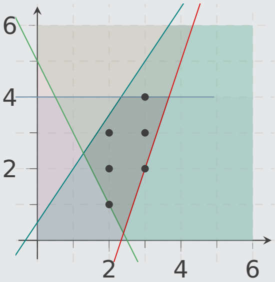

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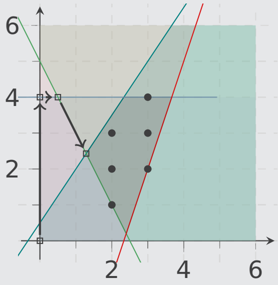

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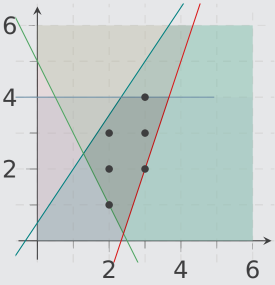

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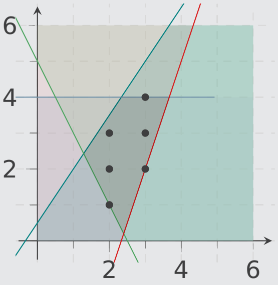

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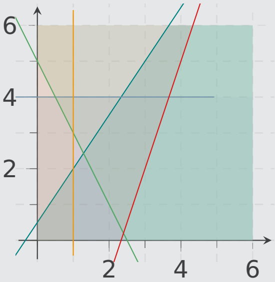

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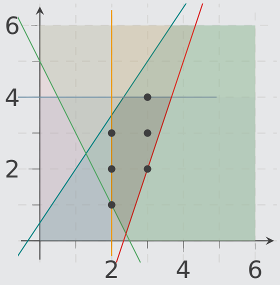

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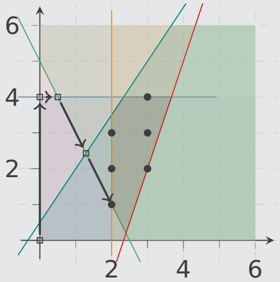

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  - $C \wedge x \geqslant 2$ satisfiable, simplex can return $(2, 1)$

Algorithm BranchAndBound(φ)

Input: LIA formula φ, a conjunction of linear inequalities
Output: unsatisfiable, or satisfying assignment

```
let res be result of deciding φ over ℝ                    ▷ e.g. by simplex
if res is unsatisfiable then
    return unsatisfiable
else if res is solution over ℤ then
    return res
else
    let x be variable assigned non-integer value q in res
    res = BranchAndBound(φ ∧ x ≤ ⌊q⌋)
    if res ≠ unsatisfiable then
        return res
    else
        return BranchAndBound(φ ∧ x ≥ ⌈q⌉)
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Example (Termination Proof of log2, Continued)

- problematic formula (satisfiable over $\mathbb{R}$)

$$\psi := x > 0 \wedge 2x' \leq x \wedge x \leq 2x' + 1 \wedge x < x' + 1 \qquad (\neg \text{ decrease})$$

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  - return unsatisfiable

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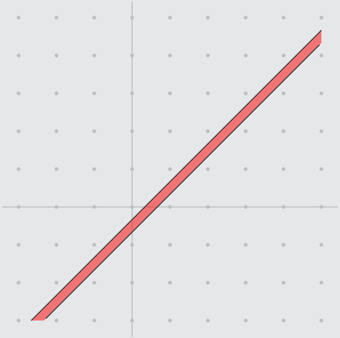
consider $\psi := 1 \leq 3x - 3y \wedge 3x - 3y \leq 2$



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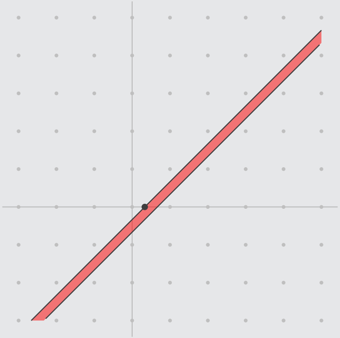

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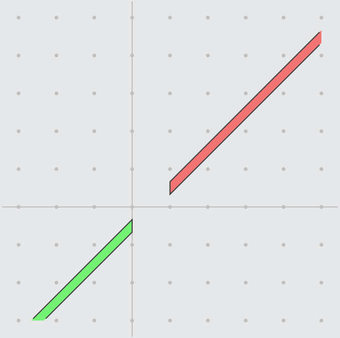

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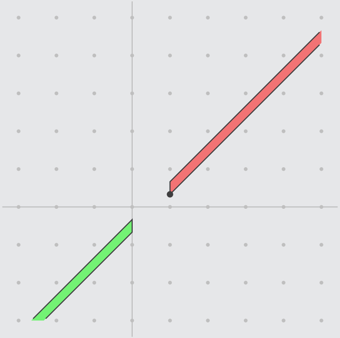

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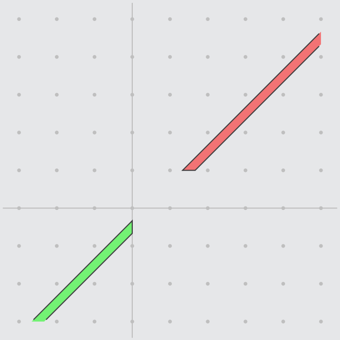

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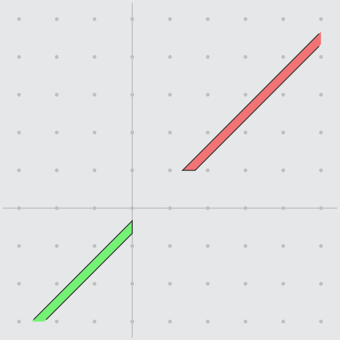

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- ... *BranchAndBound* is not terminating, since search space is unbounded

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if LIA formula ψ has solution over $\mathbb{Z}$ then it has a solution v with

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- bound is quite tight: $c \leq x_1 \wedge c \cdot x_1 \leq x_2 \wedge \ldots \wedge c \cdot x_{n-1} \leq x_n$ implies $x_n \geq c^n$

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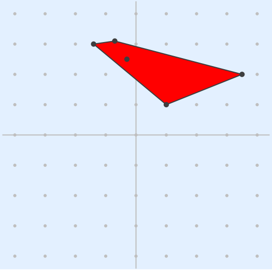
Geometric Objects

- polytope: convex hull of finite set of points X

$$hull(X) = \{\lambda_1 \vec{v}_1 + \ldots + \lambda_m \vec{v}_m \mid \{\vec{v}_1, \ldots, \vec{v}_m\} \subseteq X \wedge \lambda_1, \ldots, \lambda_m \geq 0 \wedge \sum \lambda_i = 1\}$$



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- finitely generated cone: non-negative linear combinations of finite set of vectors V

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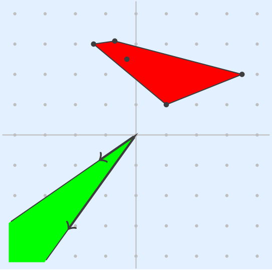

Geometric Objects

- polytope: convex hull of finite set of points X

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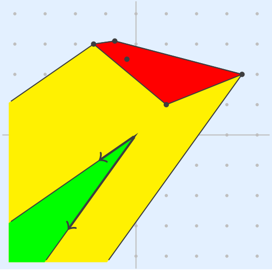
- polyhedron: polytope + finitely generated cone

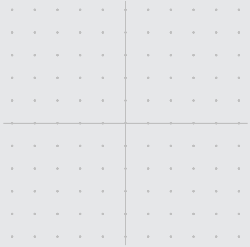
$$hull(X) + cone(V) = \{\vec{x} + \vec{v} \mid \vec{x} \in hull(X) \wedge \vec{v} \in cone(V)\}$$



More Geometric Objects

- C is polyhedral cone iff $C = \{\vec{x} \mid A\vec{x} \leq \vec{0}\}$ for some matrix A
  iff C is intersection of finitely many half-spaces

Example



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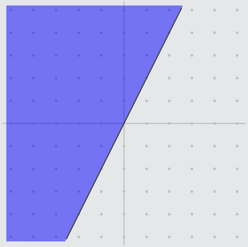

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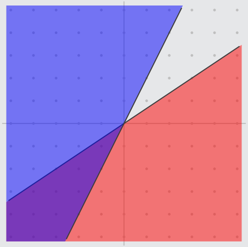

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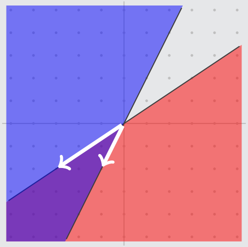

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Moreover, given X and V one can compute A and $\vec{b}$, and vice versa.

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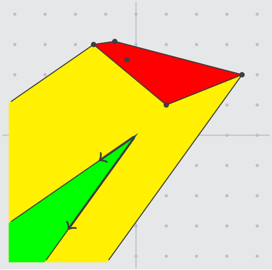

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Remark

- given ψ, one can compute X and V instead of using approximations
- however, this would be expensive: decomposition theorem requires exponentially many steps (in n, m) for input $A \in \mathbb{Z}^{m \times n}$ and $\vec{b} \in \mathbb{Z}^m$

Step 1: Conjunctive LIA Formula into Matrix Form $A\vec{x} \leq \vec{b}$

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$$x_1 > 0 \qquad 2x_2 \leq x_1 \qquad x_1 \leq 2x_2 + 1 \qquad x_1 < x_2 + 1$$

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$$\begin{pmatrix} -1 & 0 \\ -1 & 2 \\ 1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Step 3: Small Integral Solutions of Polyhedrons

- consider finite sets $X \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{Z}^n$
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$$B = \{\lambda_1 \vec{v}_1 + \ldots + \lambda_n \vec{v}_n \mid \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V \wedge 1 \geq \lambda_1, \ldots, \lambda_n \geq 0\} \subseteq cone(V)$$

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Corollary

Assume $|c| \leq b \in \mathbb{Z}$ for all entries c of all vectors in $X \cup V$.
Define $Bnd := (n+1) \cdot b$. Then

$$(hull(X) + cone(V)) \cap \mathbb{Z}^n = \emptyset$$
$$\longleftrightarrow (hull(X) + cone(V)) \cap \{-Bnd, \ldots, Bnd\}^n = \emptyset$$

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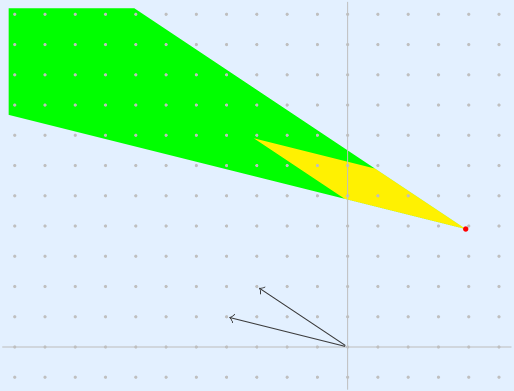

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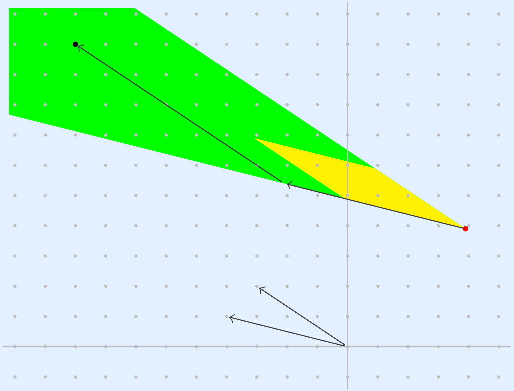

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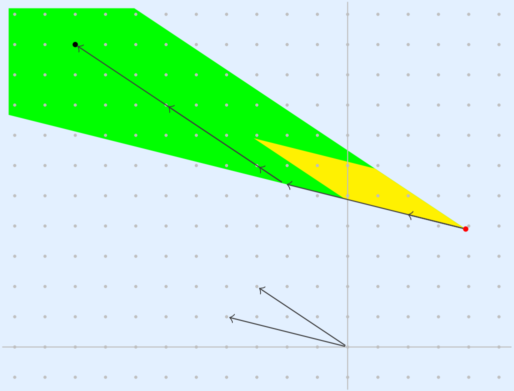

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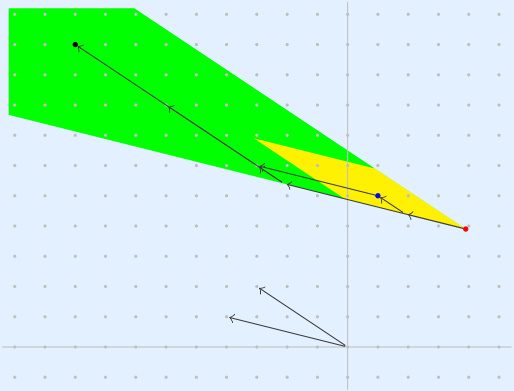

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Bounds

- the absolute values of the numbers in $X \cup V$ are all bounded by the absolute values of the numbers in C
- hence, bounds on C can be reused to bound vectors in $X \cup V$

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A cone is polyhedral iff it is finitely generated.

First direction: finitely generated implies polyhedral

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- bounds
  - each normal vector $\vec{c}$ can be computed via determinants

$\Longrightarrow$ obtain bound on numbers in $\vec{c}$ by using bounds on determinants

Example: Construction of Polyhedral Cone from Finitely Generated Cone

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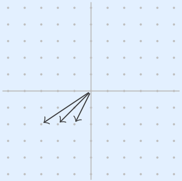

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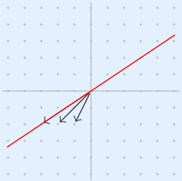

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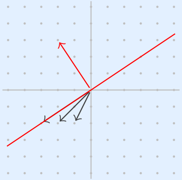

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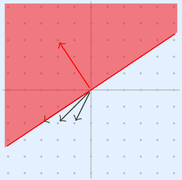

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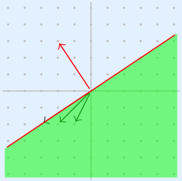

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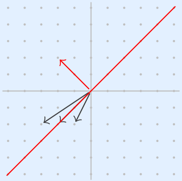

- pick $W = \{\vec{w}\}$, $\vec{w} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ and consider $span\, W$
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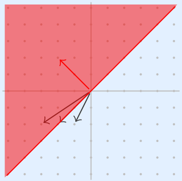
Example: Construction of Polyhedral Cone from Finitely Generated Cone



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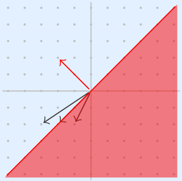

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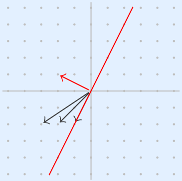

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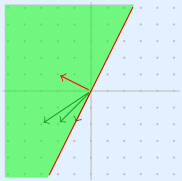

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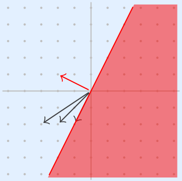

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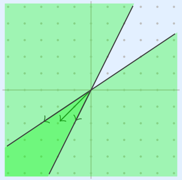

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Step 4: Theorem of Farkas, Minkowski, Weyl (bounded version)

Let $C \subseteq \mathbb{R}^n$ be a polyhedral cone, given via an integral matrix A. Let b be a bound for all matrix entries, $b \geq |A_{ij}|$. Then C is generated by a finite set of integral vectors V whose entries are at most $\pm\sqrt{(n-1)^{n-1}} \cdot b^{n-1}$.

Theorem (Hadamard's Inequality)

- *Let A be a square matrix of dimension n such that $|A_{i,j}| \leq b$ for all i, j. Then $|det(A)| \leq \sqrt{n^n} \cdot b^n$.*

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  obtain desired equality $det(A_k) = (2^k)^{2^k/2} \cdot b^{2^k}$ by induction on k:

$$det(A_{k+1}) = det(2 \cdot A_k \cdot A_k) = 2^{2^k} \cdot det(A_k)^2 = 2^{2^k} \cdot ((2^k)^{2^k/2} \cdot b^{2^k})^2 = (2^{k+1})^{2^{k+1}/2} \cdot b^{2^{k+1}}$$

Example Hadamard Matrix

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix} = 4096 = 8^4 \cdot 1^8$$

Outline

Kröning and Strichmann

- Section 5.3

Further Reading

Alexander Schrijver
Theory of linear and integer programming, Chapters 7, 16, 17, and 24
Wiley, 1998.

Important Concepts

- branch-and-bound
- cone (finitely generated or polyhedral)
- decomposition theorem for polyhedra
- Farkas–Minkowski–Weyl theorem
- Hadamard's inequality
- polyhedron
- small model property of LIA