universität innsbruck

SS 2024 lecture 10


# Constraint Solving

René Thiemann and Fabian Mitterwallner

based on a previous course by Aart Middeldorp

# Outline

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## Tightening

- convert $\sum a_i x_i < b$ to $\sum a_i x_i \leq b - 1$
- convert $\sum a_i x_i \leq b$ to $\sum \frac{a_i}{g} x_i \leq \lfloor \frac{b}{g} \rfloor$ where $g = \gcd(a_1, \ldots, a_n)$

## Unit-Cube Test

- check $A\vec{z} \leq \vec{b} - \frac{1}{2} \cdot \begin{pmatrix} |A_{11}| + \ldots + |A_{1n}| \\ \cdots \\ |A_{m1}| + \ldots + |A_{mn}| \end{pmatrix}$ by simplex
- if $\vec{z}$ is solution, then $\lfloor \vec{z} \rceil$ is integer solution of $A\vec{x} \leq \vec{b}$

## Equality Detection

- $A\vec{x} < \vec{b}$ satisfiable $\Longrightarrow$ no implied equality
- $\vec{a}_i \vec{x} < b_i$ part of minimal unsat core $\Longrightarrow$ $\vec{a}_i \vec{x} = b_i$ is implied equation
- reorder equation to $x_j = e_j$, store as part of solution, and eliminate $x_j$ elsewhere

## Diophantine Equation Solver

- return "unsat" whenever $E$ contains $\sum a_i x_i = b$ and $\gcd(a_1, \dots, a_n)$ does not divide $b$
- if $E$ contains equation $\sum a_i x_i = b$ with $|a_k| = 1$ then reorder equation to $x_k = \dots$, store as part of solution and substitute
- otherwise, pick coefficient $a_k$ with minimal absolute value
- rewrite equation as multiples of $a_k$ and remainders: $a_k(x_k + m) + r = 0$
- add equation $x_k = -m + x_t$ to solution for fresh variable $x_t$, and eliminate $x_k$

## Combined Solver

- first tighten, then perform equality detection and deletion
- tighten again, optionally detect new equalities and delete them
- try simplex as unsat criterion
- try unit-cube test as sat criterion
- decide satisfiability via branch-and-bound algorithm
- translate solution to initial set of variables via detected equalities

# Outline

1. **Summary of Previous Lecture**

2. **Bit-Vector Arithmetic**

   Theory  Flattening  Incremental Flattening

3. **Fixed-Point Arithmetic**

4. **Further Reading**

## Example (1) Change of Arithmetic: example formula $x - y > 0 \iff x > y$

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**Example (3) Hardware Verification**

Verify circuits whether they correctly implement bit-vector operations

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formulas are built according to following BNF grammar:

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## Examples

- $0_1 = 0$

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## Definition

- unsigned encoding (binary encoding) to represent non-negative integers:

$$\langle a_{k-1} \dots a_1 a_0 \rangle_u = \sum_{i=0}^{k-1} a_i 2^i$$

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- two's complement to represent negative integers (signed encoding)

$$\langle a_{k-1} \dots a_1 a_0 \rangle_s \; = \; -a_{k-1} 2^{k-1} + \sum_{i=0}^{k-2} a_k 2^i$$

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$$\langle 0000 \rangle_u = 0 = \langle 0000 \rangle_s$$

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## Remark

addition (subtraction, multiplication) is independent of signed/unsigned encoding

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## Definitions (bitwise operations)

- $\mathrm{a}_k \;\&\; \mathrm{b}_k \;=\; \mathrm{c}_k$ with $c_i = a_i \wedge b_i$ for all $0 \leqslant i < k$ bitwise and

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- $a_k \;\hat{}\; b_k \; = \; c_k$ with $c_i = a_i \oplus b_i$ for all $0 \leqslant i < k$ bitwise xor

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- $a_k \mid b_k \;=\; c_k$ with $c_i = a_i \vee b_i$ for all $0 \leqslant i < k$ — bitwise or
- $a_k \;\hat{}\; b_k \;=\; c_k$ with $c_i = a_i \oplus b_i$ for all $0 \leqslant i < k$ — bitwise xor
- $\sim a_k \;=\; c_k$ with $c_i = \neg a_i$ for all $0 \leqslant i < k$ — bitwise complement

## Examples

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- $00101011 \ \hat{} \ 01110101 = 01011110$
- $\sim 00101011 = 11010100$

## Definition (negation)

$-\, a_k \;=\; \sim a_k + 1_k$

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- $00101011 \mid 01110101 = 01111111$
- $00101011 \mathbin{\hat{}} 01110101 = 01011110$
- $\sim 00101011 = 11010100$

### Definition (negation)

$$-a_k = \sim a_k + 1_k$$

### Example

$-0101 = 1011$

## Definition (Bit-Vector Arithmetic)

formulas are built according to following BNF grammar:

$$
\begin{aligned}
\textit{formula} &::= (\neg\, \textit{formula}) \mid (\textit{formula} \wedge \textit{formula}) \mid (\textit{formula} \vee \textit{formula}) \mid \textit{atom} \\
\textit{atom} &::= \top \mid \bot \mid (\textit{term rel term}) \\
\textit{rel} &::= \; = \mid \neq \mid \geqslant_u \mid \geqslant_s \mid >_u \mid >_s \\
\textit{term} &::= (\textit{term binop term}) \mid (\textit{unop term}) \mid \textit{variable} \mid \textit{constant} \mid \textit{term}\,[i:j] \mid \\
&\phantom{::=}\ (\textit{formula} \;?\; \textit{term} : \textit{term}) \\
\textit{binop} &::= + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \& \mid \;|\; \mid \hat{} \mid :: \\
\textit{unop} &::= \sim \mid -
\end{aligned}
$$

## Notation

- variables $\mathrm{a}_k$ are vectors $a_{k-1} \dots a_1 a_0$ consisting of $k$ bits
- constant $\mathrm{n}_k$ ($\mathrm{xn}_k$) is binary representation of decimal (hexadecimal) $n$ in $k$ bits

**Definition (concatenation)**

$$\mathsf{a}_k :: \mathsf{b}_m \;=\; \mathsf{c}_{k+m} \;\text{ with }\; c_i = \begin{cases} b_i & \text{if } i < m \\ a_{i-m} & \text{otherwise} \end{cases}$$

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**Example**

$1101 :: 0010 = 11010010$

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## Definition (extraction)

$$\mathsf{a}_k[n : m] \;=\; \mathsf{c}_{n-m+1} \;\text{ with }\; c_i = a_{i+m} \text{ for all } 0 \leqslant i \leqslant n-m \qquad \text{if } k > n \geqslant m \geqslant 0$$

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## Example

$00101011[5:2] = 1010$

**Definition (if-then-else)**

$$(\varphi \; ? \; a_k \; : \; b_k) = \begin{cases} a_k & \text{if } \varphi \text{ holds} \\ b_k & \text{otherwise} \end{cases}$$

## Definition (if-then-else)

$$(\varphi \;?\; a_k : b_k) = \begin{cases} a_k & \text{if } \varphi \text{ holds} \\ b_k & \text{otherwise} \end{cases}$$

## Examples

- $a_4 + b_4 = 3_4$

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- $a_4 + b_4 = 3_4$

  satisfiable: $v(a_4) = 1_4$ and $v(b_4) = 2_4$

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## Examples

- $a_4 + b_4 = 3_4$
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- $a_4 >_u a_4 + 2_4$

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  satisfiable: $v(a_4) = 15_4$
- $a_4 >_s a_4 + 2_4$
  satisfiable: $v(a_4) = 7_4$

## Examples (cont'd)

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## Examples (cont'd)

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- $a_8 \ \& \ (a_8 - 1_8) = 0_8$
  satisfiable: $v(a_8) = 8_8$ or $v(a_8) = 16_8 = \text{x}10_8$ or $v(a_8) = \text{x}20_8$ or ...

# Outline

**Theorem**

bit-vector arithmetic is decidable if bit vectors have fixed length

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use $k$ fresh propositional variables to encode bit-vector variable $\mathsf{a}_k$

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$$\mathrm{bb}(\varphi \vee \psi) = \mathrm{bb}(\varphi) \vee \mathrm{bb}(\psi) \qquad \mathrm{bb}(\neg\varphi) = \neg\,\mathrm{bb}(\varphi) \qquad \mathrm{bb}(\top) = \top \qquad \mathrm{bb}(\bot) = \bot$$

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$$\mathsf{bb}(\varphi \wedge \psi) = \mathsf{bb}(\varphi) \wedge \mathsf{bb}(\psi)$$

$$\mathsf{bb}(t_1 \bowtie t_2) = \mathsf{bb_a}(u_1 \bowtie u_2) \wedge \varphi_1 \wedge \varphi_2 \quad \text{if } \mathsf{bb_t}(t_1) = (u_1, \varphi_1) \text{ and } \mathsf{bb_t}(t_2) = (u_2, \varphi_2)$$

## Definition (Bit-Vector Arithmetic)

formulas are built according to following BNF grammar:

$$
\begin{aligned}
\mathit{formula} &::= (\neg\, \mathit{formula}) \mid (\mathit{formula} \wedge \mathit{formula}) \mid (\mathit{formula} \vee \mathit{formula}) \mid \mathit{atom} \\
\mathit{atom} &::= \top \mid \bot \mid (\mathit{term}\ \mathit{rel}\ \mathit{term}) \\
\mathit{rel} &::= \; = \mid \neq \mid \geqslant_u \mid \geqslant_s \mid >_u \mid >_s \\
\mathit{term} &::= (\mathit{term}\ \mathit{binop}\ \mathit{term}) \mid (\mathit{unop}\ \mathit{term}) \mid \mathit{variable} \mid \mathit{constant} \mid \mathit{term}[i:j] \mid \\
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# Flattening aka Bit Blasting (cont'd)

equality

$$\mathsf{bb_a}(\mathsf{a}_k = \mathsf{b}_k) \;=\; (a_{k-1} \leftrightarrow b_{k-1}) \land \cdots \land (a_1 \leftrightarrow b_1) \land (a_0 \leftrightarrow b_0)$$

# Flattening aka Bit Blasting (cont'd)

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inequality

$$\mathrm{bb_a}(\mathrm{a}_k \neq \mathrm{b}_k) = (a_{k-1} \oplus b_{k-1}) \vee \cdots \vee (a_1 \oplus b_1) \vee (a_0 \oplus b_0)$$

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unsigned greater-than-or-equal

$$\mathsf{bb_a}(\mathsf{a}_1 \geqslant_u \mathsf{b}_1) \;=\; b_0 \to a_0$$

$$\mathsf{bb_a}(\mathsf{a}_{k+1} \geqslant_u \mathsf{b}_{k+1}) \;=\; a_k \wedge \neg b_k \vee (a_k \leftrightarrow b_k) \wedge \mathsf{bb_a}(\mathsf{a}[k-1:0] \geqslant_u \mathsf{b}[k-1:0])$$

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$$\mathrm{bb_a}(\mathsf{a}_{k+1} \geqslant_u \mathsf{b}_{k+1}) \;=\; a_k \wedge \neg b_k \vee (a_k \leftrightarrow b_k) \wedge \mathrm{bb_a}(\mathsf{a}[k-1:0] \geqslant_u \mathsf{b}[k-1:0])$$

unsigned greater-than

$$\mathrm{bb_a}(\mathsf{a}_k >_u \mathsf{b}_k) \;=\; \mathrm{bb_a}(\mathsf{a}_k \geqslant_u \mathsf{b}_k) \wedge \mathrm{bb_a}(\mathsf{a}_k \neq \mathsf{b}_k)$$

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# Flattening aka Bit Blasting (cont'd)

bitwise and

$$\mathsf{bb_t}(\mathsf{a}_k \mathbin{\&} \mathsf{b}_k) = (\mathsf{c}_k, \varphi) \quad \text{with} \quad \varphi = \bigwedge_{i=0}^{k-1} (c_i \leftrightarrow a_i \wedge b_i)$$

# Flattening aka Bit Blasting (cont'd)

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bitwise or

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bitwise xor

$$\mathrm{bb_t}(\mathrm{a}_k \,\hat{}\, \mathrm{b}_k) = (\mathrm{c}_k, \varphi) \quad \text{with} \quad \varphi = \bigwedge_{i=0}^{k-1} (c_i \leftrightarrow a_i \oplus b_i)$$

## Flattening aka Bit Blasting (cont'd)

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$$\mathrm{bb_t}(\mathrm{a}_k \mid \mathrm{b}_k) = (\mathrm{c}_k, \varphi) \quad \text{with} \quad \varphi = \bigwedge_{i=0}^{k-1} (c_i \leftrightarrow a_i \vee b_i)$$

bitwise xor

$$\mathrm{bb_t}(\mathrm{a}_k \,\hat{}\, \mathrm{b}_k) = (\mathrm{c}_k, \varphi) \quad \text{with} \quad \varphi = \bigwedge_{i=0}^{k-1} (c_i \leftrightarrow a_i \oplus b_i)$$

bitwise negation

$$\mathrm{bb_t}(\sim \mathrm{a}_k) = (\mathrm{c}_k, \varphi) \quad \text{with} \quad \varphi = \bigwedge_{i=0}^{k-1} (c_i \leftrightarrow \neg a_i)$$

## Definition (Bit-Vector Arithmetic)

formulas are built according to following BNF grammar:

$$
\begin{aligned}
\mathit{formula} &::= (\neg\, \mathit{formula}) \mid (\mathit{formula} \wedge \mathit{formula}) \mid (\mathit{formula} \vee \mathit{formula}) \mid \mathit{atom} \\
\mathit{atom} &::= \top \mid \bot \mid (\mathit{term}\ \mathit{rel}\ \mathit{term}) \\
\mathit{rel} &::= \ = \mid \neq \mid \geqslant_u \mid \geqslant_s \mid >_u \mid >_s \\
\mathit{term} &::= (\mathit{term}\ \mathit{binop}\ \mathit{term}) \mid (\mathit{unop}\ \mathit{term}) \mid \mathit{variable} \mid \mathit{constant} \mid \mathit{term}[i:j] \mid \\
&\qquad (\mathit{formula}\ ?\ \mathit{term} : \mathit{term}) \\
\mathit{binop} &::= + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \& \mid \ | \ \mid \hat{} \mid :: \\
\mathit{unop} &::= \sim \mid -
\end{aligned}
$$

## Notation

- variables a$_k$ are vectors $a_{k-1} \dots a_1 a_0$ consisting of $k$ bits
- constant n$_k$ (xn$_k$) is binary representation of decimal (hexadecimal) $n$ in $k$ bits

# Flattening aka Bit Blasting (cont'd)

concatenation

$$\mathrm{bb}_{\mathrm{t}}(\mathsf{a}_k :: \mathsf{b}_m) = (\mathsf{a}_k \mathsf{b}_m, \top)$$

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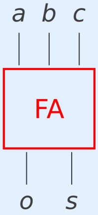
if-then-else

$$\mathsf{bb_t}(\psi \mathrel{?} \mathsf{a}_k : \mathsf{b}_k) = (\mathsf{c}_k, \varphi)$$

$$\text{with} \quad \varphi = (f \leftrightarrow \mathsf{bb}(\psi)) \wedge \bigwedge_{i=0}^{k-1} \big((f \to (c_i \leftrightarrow a_i)) \wedge (\neg f \to (c_i \leftrightarrow b_i))\big)$$

## Remark

addition is flattened using ripple carry adder




full adder

$$s = a + b + c \mod 2$$
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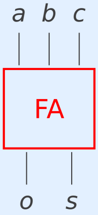


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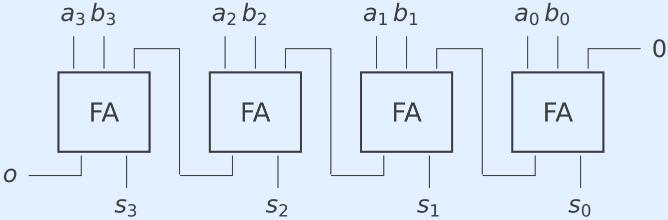

# Flattening aka Bit Blasting (cont'd)

addition

$$\mathsf{bb_t}(\mathsf{a}_k + \mathsf{b}_k) = (\mathsf{s}_k, \varphi) \quad \text{with}$$

$$\varphi = (s_0 \leftrightarrow a_0 \oplus b_0) \wedge (c_0 \leftrightarrow a_0 \wedge b_0) \wedge \bigwedge_{i=1}^{k-1} \big((s_i \leftrightarrow a_i \oplus b_i \oplus c_{i-1}) \wedge (c_i \leftrightarrow a_i \wedge b_i \vee (a_i \oplus b_i) \wedge c_{i-1})\big)$$

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## Flattening aka Bit Blasting (cont'd)

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$$\mathsf{bb_t}(\mathsf{a}_k \times \mathsf{b}_k) = \mathsf{bb_t}(\mathsf{mul}(\mathsf{a}_k, \mathsf{b}_k, 0)) \quad \text{with}$$

$$\mathsf{mul}(\mathsf{a}_k, \mathsf{b}_k, i) = \begin{cases} 0_k & \text{if } i = k \\ \mathsf{mul}(\mathsf{a}_k \ll 1_k, \mathsf{b}_k, i+1) + (b_i \mathbin{?} \mathsf{a}_k : 0_k) & \text{if } i < k \end{cases}$$

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unsigned division

$$\mathsf{bb}_t(\mathsf{a}_k \div_u \mathsf{b}_k) = (\mathsf{q}_k, \varphi) \quad \text{with}$$

$$\varphi = \mathsf{bb}(\mathsf{b}_k \neq 0_k \to (\mathsf{q}_k \times \mathsf{b}_k + \mathsf{r}_k = \mathsf{a}_k \wedge \mathsf{b}_k >_u \mathsf{r}_k \wedge \mathsf{a}_k \geqslant_u \mathsf{q}_k))$$

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bit-vector formula $\mathsf{b}_4 >_u \mathsf{a}_4 + \mathsf{b}_4 \wedge \mathsf{a}_4 \neq 10_4 \wedge \mathsf{a}_4 \,\&\, \mathsf{b}_4 = 8_4$

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# Outline

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bit-vector formula $\varphi$

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- most SAT solvers favor decisions on $a$, $b$, $c$ and thus aim to show unsatisfiability of part with multiplications

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2. use uninterpreted functions to abstract from expensive operators

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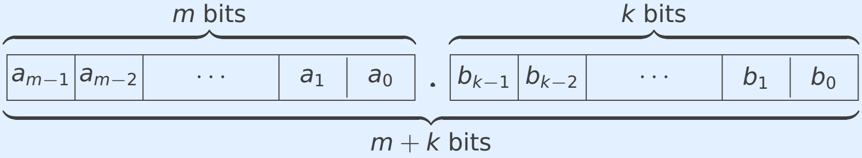

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$m$ bits (integer part), $k$ bits (fractional part), $m + k$ bits in total:

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|---|---|---|---|---|---|---|---|---|---|---|
| $a_{m-1}$ | $a_{m-2}$ | $\cdots$ | $a_1$ | $a_0$ | . | $b_{k-1}$ | $b_{k-2}$ | $\cdots$ | $b_1$ | $b_0$ |

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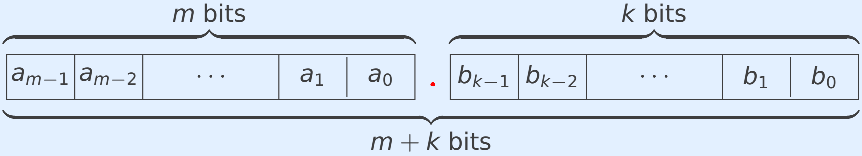


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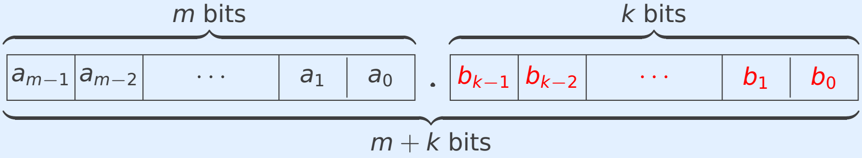


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- $b_{k-1}b_{k-2}\ldots b_1b_0$ is fractional part

## Definition

rational encoding $\langle a_{m-1} \dots a_0 \,.\, b_{k-1} \dots b_0 \rangle_r = \dfrac{\langle a_{m-1} \dots a_0 \, b_{k-1} \dots b_0 \rangle_s}{2^k}$

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## Definition (addition)

$\mathsf{a}_m \,.\, \mathsf{b}_k + \mathsf{c}_m \,.\, \mathsf{d}_k \;=\; \mathsf{e}_m \,.\, \mathsf{f}_k \quad\Longleftrightarrow\quad \langle \mathsf{a}_m \,.\, \mathsf{b}_k \rangle_r \cdot 2^k + \langle \mathsf{c}_m \,.\, \mathsf{d}_k \rangle_r \cdot 2^k = \langle \mathsf{e}_m \,.\, \mathsf{f}_k \rangle_r \cdot 2^k \mod 2^{m+k}$

## Example

$\langle 00.1 \rangle_r + \langle 00.1 \rangle_r = \langle 01.0 \rangle_r$

## Example

$\langle 00.1 \rangle_r + \langle 00.1 \rangle_r = \langle 01.0 \rangle_r \qquad \langle 000.0 \rangle_r + \langle 1.0 \rangle_r = \langle 111.0 \rangle_r$

## Example

$\langle 00.1 \rangle_r + \langle 00.1 \rangle_r = \langle 01.0 \rangle_r \qquad \langle 000.0 \rangle_r + \langle 1.0 \rangle_r = \langle 111.0 \rangle_r$

## Remark

sign-extension is needed to ensure that operands have same number of bits

**Example**

$\langle 00.1\rangle_r + \langle 00.1\rangle_r = \langle 01.0\rangle_r$ $\langle 000.0\rangle_r + \langle 1.0\rangle_r = \langle 111.0\rangle_r$

**Remark**

sign-extension is needed to ensure that operands have same number of bits

**Example**

$\langle 0.1\rangle_r \cdot \langle 1.1\rangle_r = \langle 1.11\rangle_r$

**Example**

$\langle 00.1 \rangle_r + \langle 00.1 \rangle_r = \langle 01.0 \rangle_r \qquad \langle 000.0 \rangle_r + \langle 1.0 \rangle_r = \langle 111.0 \rangle_r$

**Remark**

sign-extension is needed to ensure that operands have same number of bits

**Example**

$\langle 0.1 \rangle_r \cdot \langle 1.1 \rangle_r = \langle 1.11 \rangle_r \qquad \langle 1.1 \rangle_r \cdot \langle 1.1 \rangle_r = \langle 0.01 \rangle_r$

**Example**

$\langle 00.1 \rangle_r + \langle 00.1 \rangle_r = \langle 01.0 \rangle_r \qquad \langle 000.0 \rangle_r + \langle 1.0 \rangle_r = \langle 111.0 \rangle_r$

**Remark**

sign-extension is needed to ensure that operands have same number of bits

**Example**

$\langle 0.1 \rangle_r \cdot \langle 1.1 \rangle_r = \langle 1.11 \rangle_r \qquad \langle 1.1 \rangle_r \cdot \langle 1.1 \rangle_r = \langle 0.01 \rangle_r$

**Remark**

rounding steps may be needed to get rid of extra bits in fractional part after multiplication

# Outline

## Kröning and Strichmann

- Chapter 6
- Section A.2.3

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- Chapter 6
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## Important Concepts

- arithmetic right shift
- bit vector
- bit-vector arithmetic
- bitwise operator
- fixed-point arithmetic
- flattening
- left shift
- logical right shift
- rational encoding $\langle \cdot . \cdot \rangle_r$
- signed encoding $\langle \cdot \rangle_s$
- two's complement
- unsigned encoding $\langle \cdot \rangle_u$