universität innsbruck

SS 2024 lecture 10

Constraint Solving

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based on a previous course by Aart Middeldorp

Outline

Outline

Tightening

- convert $\sum a_i x_i < b$ to $\sum a_i x_i \leq b - 1$
- convert $\sum a_i x_i \leq b$ to $\sum \frac{a_i}{g} x_i \leq \lfloor \frac{b}{g} \rfloor$ where $g = \gcd(a_1, \ldots, a_n)$

Unit-Cube Test

- check $A\vec{z} \leq \vec{b} - \frac{1}{2} \cdot \begin{pmatrix} |A_{11}| + \ldots + |A_{1n}| \\ \cdots \\ |A_{m1}| + \ldots + |A_{mn}| \end{pmatrix}$ by simplex
- if $\vec{z}$ is solution, then $\lfloor \vec{z} \rceil$ is integer solution of $A\vec{x} \leq \vec{b}$

Equality Detection

- $A\vec{x} < \vec{b}$ satisfiable $\Longrightarrow$ no implied equality
- $\vec{a}_i\vec{x} < b_i$ part of minimal unsat core $\Longrightarrow$ $\vec{a}_i\vec{x} = b_i$ is implied equation
- reorder equation to $x_j = e_j$, store as part of solution, and eliminate x_j elsewhere

Diophantine Equation Solver

- return "unsat" whenever E contains $\sum a_i x_i = b$ and $\gcd(a_1, \ldots, a_n)$ does not divide b
- if E contains equation $\sum a_i x_i = b$ with $|a_k| = 1$ then reorder equation to $x_k = \ldots$, store as part of solution and substitute
- otherwise, pick coefficient a_k with minimal absolute value
- rewrite equation as multiples of a_k and remainders: $a_k(x_k + m) + r = 0$
- add equation $x_k = -m + x_t$ to solution for fresh variable x_t, and eliminate x_k

Combined Solver

- first tighten, then perform equality detection and deletion
- tighten again, optionally detect new equalities and delete them
- try simplex as unsat criterion
- try unit-cube test as sat criterion
- decide satisfiability via branch-and-bound algorithm
- translate solution to initial set of variables via detected equalities

Outline

Example (1) Change of Arithmetic: example formula $x - y > 0 \iff x > y$

- is valid over the integers
- is not valid if x and y are interpreted as finite-width bit vectors (due to overflows)

Example (2) Actual Program Behavior, e.g., for C

```
unsigned char number = 200;
number = number + 100;
printf("sum: %d\n", number);
```

prints 44 if variables of type unsigned char are represented by 8 bits:

$$\overset{200}{11001000} + \overset{100}{01100100} = \overset{44}{00101100}$$

Example (3) Hardware Verification

Verify circuits whether they correctly implement bit-vector operations

Definition (Bit-Vector Arithmetic)

formulas are built according to following BNF grammar:

$$\begin{aligned}
\mathit{formula} &::= (\neg\, \mathit{formula}) \mid (\mathit{formula} \wedge \mathit{formula}) \mid (\mathit{formula} \vee \mathit{formula}) \mid \mathit{atom} \\
\mathit{atom} &::= \top \mid \bot \mid (\mathit{term}\ \mathit{rel}\ \mathit{term}) \\
\mathit{rel} &::= = \mid \neq \mid \geqslant_u \mid \geqslant_s \mid >_u \mid >_s \\
\mathit{term} &::= (\mathit{term}\ \mathit{binop}\ \mathit{term}) \mid (\mathit{unop}\ \mathit{term}) \mid \mathit{variable} \mid \mathit{constant} \mid \mathit{term}[i:j] \mid \\
&\quad\ (\mathit{formula}\ ?\ \mathit{term} : \mathit{term}) \\
\mathit{binop} &::= + \mid - \mid \times \mid \div_u \mid \div_s \mid \%_u \mid \%_s \mid \ll \mid \gg_u \mid \gg_s \mid \&\ \mid\ \mid \mid\ \hat{}\ \mid :: \\
\mathit{unop} &::= \sim \mid -
\end{aligned}$$

Notation

- variables a_k are vectors $a_{k-1} \ldots a_1 a_0$ consisting of k bits
- constant n_k (xn_k) is binary representation of decimal (hexadecimal) n in k bits

Examples

- $0_1 = 0 \quad 3_2 = 11 \quad 10_4 = 1010 \quad 1024_{32} = 00000000000000000000010000000000$
- $x0_4 = 0000 \quad xa_4 = 1010 \quad xb0_8 = 10110000 \quad x11cf_{16} = 0001000111001111$

Definition

- unsigned encoding (binary encoding) to represent non-negative integers:

$$\langle a_{k-1} \dots a_1 a_0 \rangle_u \;=\; \sum_{i=0}^{k-1} a_i 2^i$$

- two's complement to represent negative integers (signed encoding)

$$\langle a_{k-1} \dots a_1 a_0 \rangle_s \;=\; -a_{k-1} 2^{k-1} + \sum_{i=0}^{k-2} a_k 2^i$$

Example

$\langle 0000 \rangle_u = 0 = \langle 0000 \rangle_s \qquad \langle 1010 \rangle_u = 10 \neq -6 = \langle 1010 \rangle_s \qquad \langle 10110000 \rangle_s = -80$

Example

- $1001 + 0011 = 1100$
  unsigned: $9 + 3 = 12$
  signed: $(-7) + 3 = -4$
- $1011 + 0110 = 0001$
  unsigned: $11 + 6 = 1 \bmod 16$
  signed: $(-5) + 6 = 1$

Remark

addition (subtraction, multiplication) is independent of signed/unsigned encoding

Definition (left shift)

$$a_k \ll b_k \;=\; c_k \;\text{ with }\; c_i = \begin{cases} a_{i-\langle b_k \rangle_u} & \text{if } i - \langle b_k \rangle_u \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Example

$00101011 \ll 00000100 = 10110000$

Definition (logical right shift)

$$a_k \gg_u b_k \;=\; c_k \;\text{ with }\; c_i = \begin{cases} a_{i+\langle b_k \rangle_u} & \text{if } i + \langle b_k \rangle_u < k \\ 0 & \text{otherwise} \end{cases}$$

Example

$00101011 \gg_u 00000100 = 00000010$

Definition (arithmetic right shift)

$$a_k \gg_s b_k \;=\; c_k \;\text{ with }\; c_i = \begin{cases} a_{i+\langle b_k \rangle_u} & \text{if } i + \langle b_k \rangle_u < k \\ a_{k-1} & \text{otherwise} \end{cases}$$

Example

$00101011 \gg_s 00000100 = 00000010 \qquad 10101011 \gg_s 00000100 = 11111010$

Definitions (bitwise operations)

- $a_k \;\&\; b_k = c_k$ with $c_i = a_i \wedge b_i$ for all $0 \leqslant i < k$ bitwise and
- $a_k \mid b_k = c_k$ with $c_i = a_i \vee b_i$ for all $0 \leqslant i < k$ bitwise or
- $a_k \;\hat{}\; b_k = c_k$ with $c_i = a_i \oplus b_i$ for all $0 \leqslant i < k$ bitwise xor
- $\sim a_k \;= c_k$ with $c_i = \neg a_i$ for all $0 \leqslant i < k$ bitwise complement

Examples

- $00101011 \;\&\; 01110101 = 00100001$
- $00101011 \mid 01110101 = 01111111$
- $00101011 \;\hat{}\; 01110101 = 01011110$
- $\sim 00101011 = 11010100$

Definition (negation)

$-\,a_k \;=\; \sim a_k + 1_k$

Example

$-\,0101 = 1011$

Definition (concatenation)

$a_k :: b_m \;=\; c_{k+m}$ with $c_i = \begin{cases} b_i & \text{if } i < m \\ a_{i-m} & \text{otherwise} \end{cases}$

Example

$1101 :: 0010 = 11010010$

Definition (extraction)

$a_k[n:m] \;=\; c_{n-m+1}$ with $c_i = a_{i+m}$ for all $0 \leqslant i \leqslant n-m$ if $k > n \geqslant m \geqslant 0$

Example

$00101011[5:2] = 1010$

Definition (if-then-else)

$(\varphi \;?\; a_k \;:\; b_k) \;=\; \begin{cases} a_k & \text{if } \varphi \text{ holds} \\ b_k & \text{otherwise} \end{cases}$

Examples

- $a_4 + b_4 = 3_4$
  satisfiable: $v(a_4) = 1_4$ and $v(b_4) = 2_4$
- $a_4 >_u a_4 + 2_4$
  satisfiable: $v(a_4) = 15_4$
- $a_4 >_s a_4 + 2_4$
  satisfiable: $v(a_4) = 7_4$

Examples (cont'd)

- $a_4 \geqslant_u b_4 \wedge \neg(a_4 \geqslant_s b_4)$
  satisfiable: $v(a_4) = 8_4$ and $v(b_4) = 0_4$
- $-a_4 = a_4 \wedge \neg(a_4 = 0_4)$
  satisfiable: $v(a_4) = -8_4 = 8_4$
- $a_4 \ll 2_4 = 12_4$
  satisfiable: $v(a_4) = 11_4$
- $a_4[1:0] :: a_4[3:2] = 2_4 \wedge b_4[2:0] = 7_3$
  satisfiable: $v(a_4) = 8_4$ and $v(b_4) = 15_4$
- $a_8 \div_u b_8 = a_8 \gg_u 1_8$
  satisfiable: $v(a_8) = 8_8$ and $v(b_8) = 2_8$
- $a_8 \;\&\; (a_8 - 1_8) = 0_8$
  satisfiable: $v(a_8) = 8_8$ or $v(a_8) = 16_8 = \text{x}10_8$ or $v(a_8) = \text{x}20_8$ or ...

Outline

Theorem

bit-vector arithmetic is decidable if bit vectors have fixed length

Basic Idea

use k fresh propositional variables to encode bit-vector variable a_k

Flattening aka Bit Blasting

input: formula φ in bit-vector arithmetic

output: equisatisfiable propositional formula $\mathrm{bb}(\varphi)$

$$\mathrm{bb}(\varphi \vee \psi) = \mathrm{bb}(\varphi) \vee \mathrm{bb}(\psi) \qquad \mathrm{bb}(\neg\varphi) = \neg\,\mathrm{bb}(\varphi) \qquad \mathrm{bb}(\top) = \top \qquad \mathrm{bb}(\bot) = \bot$$

$$\mathrm{bb}(\varphi \wedge \psi) = \mathrm{bb}(\varphi) \wedge \mathrm{bb}(\psi)$$

$$\mathrm{bb}(t_1 \bowtie t_2) = \mathrm{bb_a}(u_1 \bowtie u_2) \wedge \varphi_1 \wedge \varphi_2 \quad \text{if } \mathrm{bb_t}(t_1) = (u_1, \varphi_1) \text{ and } \mathrm{bb_t}(t_2) = (u_2, \varphi_2)$$

Flattening aka Bit Blasting (cont'd)

equality

$$\mathrm{bb_a}(\mathsf{a}_k = \mathsf{b}_k) = (a_{k-1} \leftrightarrow b_{k-1}) \wedge \cdots \wedge (a_1 \leftrightarrow b_1) \wedge (a_0 \leftrightarrow b_0)$$

inequality

$$\mathrm{bb_a}(\mathsf{a}_k \neq \mathsf{b}_k) = (a_{k-1} \oplus b_{k-1}) \vee \cdots \vee (a_1 \oplus b_1) \vee (a_0 \oplus b_0)$$

unsigned greater-than-or-equal

$$\mathrm{bb_a}(\mathsf{a}_1 \geqslant_u \mathsf{b}_1) = b_0 \rightarrow a_0$$
$$\mathrm{bb_a}(\mathsf{a}_{k+1} \geqslant_u \mathsf{b}_{k+1}) = a_k \wedge \neg b_k \vee (a_k \leftrightarrow b_k) \wedge \mathrm{bb_a}(\mathsf{a}[k-1:0] \geqslant_u \mathsf{b}[k-1:0])$$

unsigned greater-than

$$\mathrm{bb_a}(\mathsf{a}_k >_u \mathsf{b}_k) = \mathrm{bb_a}(\mathsf{a}_k \geqslant_u \mathsf{b}_k) \wedge \mathrm{bb_a}(\mathsf{a}_k \neq \mathsf{b}_k)$$

Flattening aka Bit Blasting (cont'd)

bitwise and

$$\mathrm{bb_t}(\mathsf{a}_k \,\&\, \mathsf{b}_k) = (\mathsf{c}_k, \varphi) \quad \text{with} \quad \varphi = \bigwedge_{i=0}^{k-1} (c_i \leftrightarrow a_i \wedge b_i)$$

bitwise or

$$\mathrm{bb_t}(\mathsf{a}_k \mid \mathsf{b}_k) = (\mathsf{c}_k, \varphi) \quad \text{with} \quad \varphi = \bigwedge_{i=0}^{k-1} (c_i \leftrightarrow a_i \vee b_i)$$

bitwise xor

$$\mathrm{bb_t}(\mathsf{a}_k \,\hat{}\, \mathsf{b}_k) = (\mathsf{c}_k, \varphi) \quad \text{with} \quad \varphi = \bigwedge_{i=0}^{k-1} (c_i \leftrightarrow a_i \oplus b_i)$$

bitwise negation

$$\mathrm{bb_t}(\sim \mathsf{a}_k) = (\mathsf{c}_k, \varphi) \quad \text{with} \quad \varphi = \bigwedge_{i=0}^{k-1} (c_i \leftrightarrow \neg a_i)$$

Flattening aka Bit Blasting (cont'd)

concatenation

$$\mathrm{bb_t}(\mathsf{a}_k :: \mathsf{b}_m) \;=\; (a_k b_m, \top)$$

extraction

$$\mathrm{bb_t}(\mathsf{a}_k[n:m]) \;=\; (a_n \ldots a_m, \top)$$

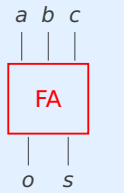
if-then-else

$$\mathrm{bb_t}(\psi \;?\; \mathsf{a}_k : \mathsf{b}_k) \;=\; (\mathsf{c}_k, \varphi)$$

$$\text{with} \quad \varphi \;=\; (f \leftrightarrow \mathrm{bb}(\psi)) \wedge \bigwedge_{i=0}^{k-1} \left((f \rightarrow (c_i \leftrightarrow a_i)) \wedge (\neg f \rightarrow (c_i \leftrightarrow b_i))\right)$$

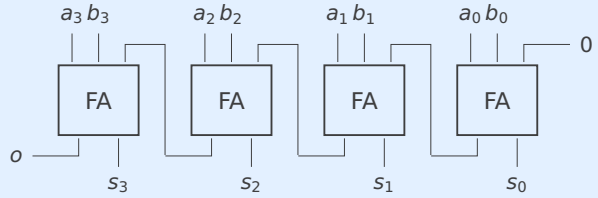
Remark

addition is flattened using ripple carry adder



full adder $\quad s = a + b + c \mod 2$, $\; o = a + b + c \operatorname{div} 2$

4 bit ripple carry adder



Flattening aka Bit Blasting (cont'd)

addition

$$\mathrm{bb_t}(\mathsf{a}_k + \mathsf{b}_k) \;=\; (\mathsf{s}_k, \varphi) \quad \text{with}$$

$$\varphi \;=\; (s_0 \leftrightarrow a_0 \oplus b_0) \wedge (c_0 \leftrightarrow a_0 \wedge b_0) \wedge \bigwedge_{i=1}^{k-1} \left((s_i \leftrightarrow a_i \oplus b_i \oplus c_{i-1}) \wedge (c_i \leftrightarrow a_i \wedge b_i \vee (a_i \oplus b_i) \wedge c_{i-1})\right)$$

unary minus

$$\mathrm{bb_t}(-\mathsf{a}_k) \;=\; \mathrm{bb_t}(\sim \mathsf{a}_k + 1_k)$$

subtraction

$$\mathrm{bb_t}(\mathsf{a}_k - \mathsf{b}_k) \;=\; \mathrm{bb_t}(\mathsf{a}_k + (-\mathsf{b}_k))$$

Flattening aka Bit Blasting (cont'd)

multiplication

$$\mathrm{bb_t}(\mathsf{a}_k \times \mathsf{b}_k) \;=\; \mathrm{bb_t}(\mathrm{mul}(\mathsf{a}_k, \mathsf{b}_k, 0)) \quad \text{with}$$

$$\mathrm{mul}(\mathsf{a}_k, \mathsf{b}_k, i) \;=\; \begin{cases} 0_k & \text{if } i = k \\ \mathrm{mul}(\mathsf{a}_k \ll 1_k, \mathsf{b}_k, i+1) + (b_i \;?\; \mathsf{a}_k : 0_k) & \text{if } i < k \end{cases}$$

unsigned division

$$\mathrm{bb_t}(\mathsf{a}_k \div_u \mathsf{b}_k) \;=\; (\mathsf{q}_k, \varphi) \quad \text{with}$$

$$\varphi \;=\; \mathrm{bb}(\mathsf{b}_k \neq 0_k \rightarrow (\mathsf{q}_k \times \mathsf{b}_k + \mathsf{r}_k = \mathsf{a}_k \;\wedge\; \mathsf{b}_k >_u \mathsf{r}_k \;\wedge\; \mathsf{a}_k \geqslant_u \mathsf{q}_k))$$

SMT-LIB 2 Format for BV

bit-vector formula $b_4 >_u a_4 + b_4 \wedge a_4 \neq 10_4 \wedge a_4 \,\&\, b_4 = 8_4$

```
(declare-const a (_ BitVec 4))
(declare-const b (_ BitVec 4))
(assert (bvugt b (bvadd a b)))
(assert (not (= a #xa)))
(assert (= (bvand a b) #b1000))
(check-sat)
```

- `(_ BitVec k)` is sort of bit vectors of width k
- `#xa` is constant in hexidecimal notation
- `#b1000` is constant in binary notation
- `bvadd`, `bvsub`, `bvmul` are arithmetic operators
- `bvudiv` and `bvsdiv` are unsigned/signed division
- `bvugt` and `bvsgt` are unsigned/signed greater-than
- `bvshl`, `bvlshr`, `bvashr` are shift operators
- `bvand`, `bvor`, `bvnot` are bitwise logical operators
- ...

Outline

Remark

bit-vector formulas with multiplication (division, remainder) are very hard to solve due to decision heuristics of state-of-the-art SAT solvers

Example

bit-vector formula φ

$$a_k \times b_k = c_k \;\wedge\; b_k \times a_k \neq c_k \;\wedge\; d_k >_u e_k \;\wedge\; e_k >_u d_k$$

results in propositional CNF formula with about 11000 variables for $k = 32$

- φ is unsatisfiable
- most SAT solvers favor decisions on a, b, c and thus aim to show unsatisfiability of part with multiplications

Solutions

1. incremental flattening
   - omit $\mathrm{bb}_t(\cdot)$ constraints for expensive operators from bit-vector formula φ
   - if SAT solver returns unsatisfiable then φ is unsatisfiable
   - if SAT solver returns satisfiable then check whether obtained model satisfies omitted constraints (after extending assignment)

     yes $\implies$ φ is satisfiable

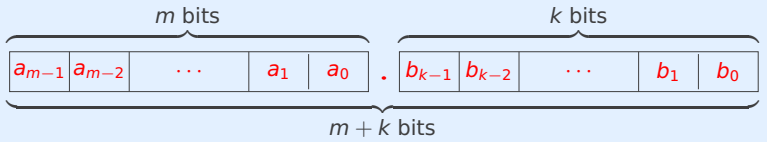
     no $\implies$ add constraints for some violated $\mathrm{bb}_t(\cdot)$ part and repeat
2. use uninterpreted functions to abstract from expensive operators

Outline

Remarks

- many applications require arithmetic on rational numbers
- high-end microprocessors offer support for floating-point arithmetic
- fixed-point arithmetic is reasonable compromise between accuracy and complexity of floating-point arithmetic, e.g., used for controllers in vehicles or for currencies in financial software

Representation



- $a_{m-1}a_{m-2}\ldots a_1a_0$ is integer part (magnitude)
- $.$ is radix point
- $b_{k-1}b_{k-2}\ldots b_1b_0$ is fractional part

Definition

rational encoding $\langle a_{m-1}\ldots a_0 \,.\, b_{k-1}\ldots b_0\rangle_r \;=\; \dfrac{\langle a_{m-1}\ldots a_0 b_{k-1}\ldots b_0\rangle_s}{2^k}$

Examples

- $\langle 0.10\rangle_r = 0.5$
- $\langle 0.01\rangle_r = 0.25$
- $\langle 01.1\rangle_r = 1.5$
- $\langle 11111111.1\rangle_r = -0.5$
- $\langle 0000.0101\rangle_r = 0.3125 < \frac{1}{3} < 0.375 = \langle 0000.0110\rangle_r$

Definition (addition)

$a_m.b_k + c_m.d_k \;=\; e_m.f_k \iff \langle a_m.b_k\rangle_r \cdot 2^k + \langle c_m.d_k\rangle_r \cdot 2^k = \langle e_m.f_k\rangle_r \cdot 2^k \mod 2^{m+k}$

Example

$\langle 00.1\rangle_r + \langle 00.1\rangle_r = \langle 01.0\rangle_r \qquad \langle 000.0\rangle_r + \langle 1.0\rangle_r = \langle 111.0\rangle_r$

Remark

sign-extension is needed to ensure that operands have same number of bits

Example

$\langle 0.1\rangle_r \cdot \langle 1.1\rangle_r = \langle 1.11\rangle_r \qquad \langle 1.1\rangle_r \cdot \langle 1.1\rangle_r = \langle 0.01\rangle_r$

Remark

rounding steps may be needed to get rid of extra bits in fractional part after multiplication

Outline

Kröning and Strichmann

- Chapter 6
- Section A.2.3

Important Concepts

- arithmetic right shift
- bit vector
- bit-vector arithmetic
- bitwise operator
- fixed-point arithmetic
- flattening
- left shift
- logical right shift
- rational encoding $\langle \cdot . . \cdot \rangle_r$
- signed encoding $\langle \cdot \rangle_s$
- two's complement
- unsigned encoding $\langle \cdot \rangle_u$