universität innsbruck

Constraint Solving

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based on a previous course by Aart Middeldorp

Outline

1. **Nelson–Oppen Combination Method**
2. **Quantified Boolean Formulas**
3. **PSPACE-Completeness of QBF**
4. **Further Reading**

Current State

- SMT solver for various theories

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  - equality logic — equality graphs
  - EUF (equality with uninterpreted functions) — congruence closure
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  - BV (bit-vector arithmetic) — flattening

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- upcoming: combination method which communicate via equalities
- requirement for presented combination algorithm: theories must be stably infinite

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- axioms T: set of sentences in first-order logic in which only function and predicate symbols of Σ appear

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theory combination $T_1 \oplus T_2$ of two theories

- T_1 over signature Σ_1
- T_2 over signature Σ_2

has signature $\Sigma_1 \cup \Sigma_2$ and axioms $T_1 \cup T_2$

Outline

Example

combination of linear arithmetic and uninterpreted functions:

$$x \geqslant y \ \wedge \ y - z \geqslant x \ \wedge \ \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \ \wedge \ z \geqslant 0$$

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- $\Sigma_1 \cap \Sigma_2 = \{=\}$
- T_1-satisfiability of quantifier-free Σ_1-formulas is decidable
- T_2-satisfiability of quantifier-free Σ_2-formulas is decidable

Nelson–Oppen Method: Nondeterministic Version

input: quantifier-free conjunction φ in theory combination $T_1 \oplus T_2$

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formula φ in combination of LIA and EUF:

$$1 \leqslant x \ \wedge\ x \leqslant 2 \ \wedge\ \mathsf{f}(x) \neq \mathsf{f}(1) \ \wedge\ \mathsf{f}(x) \neq \mathsf{f}(2)$$

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   - arrangement $\alpha(V, E)$ is formula

$$\left(\bigwedge_{x\,E\,y} x = y \right) \;\wedge\; \left(\bigwedge_{\neg(x\,E\,y)} x \neq y \right)$$

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- $V = \{x, y, z\}$
- 5 different equivalence relations E:
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2. guess and check
   - V is set of shared variables in φ_1 and φ_2
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   - if $\varphi_1 \wedge \alpha(V, E)$ is T_1-satisfiable and $\varphi_2 \wedge \alpha(V, E)$ is T_2-satisfiable then return satisfiable else return unsatisfiable

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theory T is convex if for every quantifier-free conjunctive formula F and $n \geq 1$:

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purification

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unsatisfiable

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   if $\varphi_1 \wedge \mathcal{E}$ is T_1-unsatisfiable then return unsatisfiable

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Example

combination of LIA and EUF: $1 \leqslant x \wedge x \leqslant 2 \wedge \mathsf{f}(x) \neq \mathsf{f}(1) \wedge \mathsf{f}(x) \neq \mathsf{f}(2)$

purification

$$\varphi_1: \quad 1 \leqslant x \wedge x \leqslant 2 \wedge w_1 = 1 \wedge w_2 = 2$$

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implied equalities between shared variables (first case)

$$\mathcal{E}: \ x = w_1$$

test satisfiability of $\varphi_1 \ \wedge \ \mathcal{E} \quad \Longrightarrow \quad x = w_1 \ \vee \ x = w_2$

case split: $x = w_1$ or $x = w_2$

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test satisfiability of $\varphi_2 \wedge \mathcal{E} \implies \bot$

case split: $x = w_1$ or $x = w_2$

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implied equalities between shared variables (second case)

$$\mathcal{E}: \ x = w_2$$

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Requirement of Convex Theories for Deterministic Nelson–Oppen

- combine LIA (non-convex) + EUF (convex)
- problem with deterministic algorithm
  - $\underbrace{1 \leqslant x \wedge x \leqslant 2 \wedge y_1 = 1 \wedge y_2 = 2}_{\varphi_{LIA}} \wedge \underbrace{f(x) = a \wedge f(y_1) \neq a \wedge f(y_2) \neq a}_{\varphi_{EUF}}$

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Requirement of Stably Infinite Theories for Nelson–Oppen

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- problem with non-deterministic Nelson–Oppen algorithm
  - $\underbrace{x_4 \geqslant_u 0}_{\varphi_{BV}} \wedge \underbrace{\left(\bigwedge_{1 \leqslant i \leqslant 16} x_4 \neq a_i\right) \wedge \left(\bigwedge_{1 \leqslant i < j \leqslant 16} a_i \neq a_j\right)}_{\varphi_{EUF}}$

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  - Nelson–Oppen reports satisfiability (only one trivial arrangement $\{x_4\}$, but the 4-bit vector x_4 cannot be different to 16 different constants $a_1, \dots, a_{16}$)

Outline

Current State

- SMT solving for various (combinations of) theories
- SMT: implicit existential quantification
- upcoming lectures: integrate quantifiers
- this lecture considers simplest case: **Q**uantified **B**oolean **F**ormulas

Definition (QBF Syntax)

$$\varphi \ ::= \ \bot \ \mid \ \top \ \mid \ x \ \mid \ \neg\varphi \ \mid \ \varphi \wedge \varphi \ \mid \ \varphi \vee \varphi \ \mid \ \varphi \rightarrow \varphi \ \mid \ \forall x.\, \varphi \ \mid \ \exists x.\, \varphi$$

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Definition (QBF Semantics)

extension of propositional logic case:

$$v(\forall x.\, \varphi) = \begin{cases} \mathsf{T} & \text{if } v[x/\mathsf{F}](\varphi) = \mathsf{T} \text{ and } v[x/\mathsf{T}](\varphi) = \mathsf{T} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

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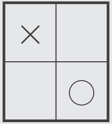
Definition

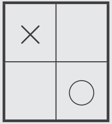
prenex normal form: $Q_1 x_1 \dots Q_n x_n.\, \varphi$ with $Q_i \in \{\forall, \exists\}$ for all $1 \leqslant i \leqslant n$ and φ quantifier-free

Example (2 × 2 Tic-Tac-Toe)



- two players (× and ○)

Example (2 $\times$ 2 Tic-Tac-Toe)



- two players ($\times$ and $\bigcirc$)
- first player ($\times$) has winning strategy

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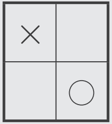

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$$\exists \times \text{ move } \quad (\times \text{ wins or}$$

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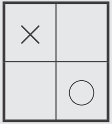

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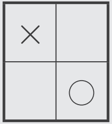

- two players ($\times$ and $\bigcirc$)
- first player ($\times$) has winning strategy

$$
\begin{array}{ll}
\exists \times \text{ move} & (\times \text{ wins or} \\
\forall \bigcirc \text{ move} & (\bigcirc \text{ does not win and} \\
\exists \times \text{ move} & \quad \times \text{ wins}))
\end{array}
$$

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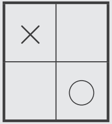

- two players (× and ○)
- first player (×) has winning strategy

$$\begin{array}{ll} \exists \times \text{ move} & (\times \text{ wins or} \\ \forall \bigcirc \text{ move} & (\bigcirc \text{ does not win and} \\ \exists \times \text{ move} & \quad \times \text{ wins})) \end{array}$$

- expressible in QBF

Lemma

every quantified formula can be transformed into equivalent prenex normal form

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Proof

1. eliminate all propositional connectives other than $\vee$, $\wedge$, $\neg$
2. rename all bound variables such that every quantifier binds unique variable
3. push propositional connectives through quantifiers:

$$
\begin{aligned}
\neg\forall x.\,\varphi &\iff \exists x.\,\neg\varphi & \neg\exists x.\,\varphi &\iff \forall x.\,\neg\varphi \\
(\forall x.\,\varphi)\wedge\psi &\iff \forall x.\,\varphi\wedge\psi & \varphi\wedge\forall x.\,\psi &\iff \forall x.\,\varphi\wedge\psi \\
(\exists x.\,\varphi)\wedge\psi &\iff \exists x.\,\varphi\wedge\psi & \varphi\wedge\exists x.\,\psi &\iff \exists x.\,\varphi\wedge\psi \\
(\forall x.\,\varphi)\vee\psi &\iff \forall x.\,\varphi\vee\psi & \varphi\vee\forall x.\,\psi &\iff \forall x.\,\varphi\vee\psi \\
(\exists x.\,\varphi)\vee\psi &\iff \exists x.\,\varphi\vee\psi & \varphi\vee\exists x.\,\psi &\iff \exists x.\,\varphi\vee\psi
\end{aligned}
$$

$$\neg\forall x.\,\varphi \iff \exists x.\,\neg\varphi \qquad \neg\exists x.\,\varphi \iff \forall x.\,\neg\varphi$$
$$(\forall x.\,\varphi)\wedge\psi \iff \forall x.\,\varphi\wedge\psi \qquad \varphi\wedge\forall x.\,\psi \iff \forall x.\,\varphi\wedge\psi$$
$$(\exists x.\,\varphi)\wedge\psi \iff \exists x.\,\varphi\wedge\psi \qquad \varphi\wedge\exists x.\,\psi \iff \exists x.\,\varphi\wedge\psi$$
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$$(\exists x.\,\varphi)\vee\psi \iff \exists x.\,\varphi\vee\psi \qquad \varphi\vee\exists x.\,\psi \iff \exists x.\,\varphi\vee\psi$$

Example

- $\neg\exists x.\,\neg((\exists y.\,(y\rightarrow x)\wedge(\neg x\vee y))\wedge\neg(\forall y.\,y\wedge x\vee\neg x\wedge\neg y))$

$$\neg\forall x.\,\varphi \iff \exists x.\,\neg\varphi \qquad \neg\exists x.\,\varphi \iff \forall x.\,\neg\varphi$$

$$(\forall x.\,\varphi)\land\psi \iff \forall x.\,\varphi\land\psi \qquad \varphi\land\forall x.\,\psi \iff \forall x.\,\varphi\land\psi$$

$$(\exists x.\,\varphi)\land\psi \iff \exists x.\,\varphi\land\psi \qquad \varphi\land\exists x.\,\psi \iff \exists x.\,\varphi\land\psi$$

$$(\forall x.\,\varphi)\lor\psi \iff \forall x.\,\varphi\lor\psi \qquad \varphi\lor\forall x.\,\psi \iff \forall x.\,\varphi\lor\psi$$

$$(\exists x.\,\varphi)\lor\psi \iff \exists x.\,\varphi\lor\psi \qquad \varphi\lor\exists x.\,\psi \iff \exists x.\,\varphi\lor\psi$$

Example

- $\neg\exists x.\,\neg((\exists y.\,(y \to x)\land(\neg x\lor y))\land\neg(\forall y.\,y\land x\lor\neg x\land\neg y))$
- $\neg\exists x.\,\neg((\exists y.\,(\neg y \lor x)\land(\neg x\lor y))\land\neg(\forall y.\,y\land x\lor\neg x\land\neg y))$

$$\neg\forall x.\,\varphi \iff \exists x.\,\neg\varphi \qquad \neg\exists x.\,\varphi \iff \forall x.\,\neg\varphi$$

$$(\forall x.\,\varphi) \wedge \psi \iff \forall x.\,\varphi \wedge \psi \qquad \varphi \wedge \forall x.\,\psi \iff \forall x.\,\varphi \wedge \psi$$

$$(\exists x.\,\varphi) \wedge \psi \iff \exists x.\,\varphi \wedge \psi \qquad \varphi \wedge \exists x.\,\psi \iff \exists x.\,\varphi \wedge \psi$$

$$(\forall x.\,\varphi) \vee \psi \iff \forall x.\,\varphi \vee \psi \qquad \varphi \vee \forall x.\,\psi \iff \forall x.\,\varphi \vee \psi$$

$$(\exists x.\,\varphi) \vee \psi \iff \exists x.\,\varphi \vee \psi \qquad \varphi \vee \exists x.\,\psi \iff \exists x.\,\varphi \vee \psi$$

Example

- $\neg\exists x.\,\neg((\exists y.\,(y \rightarrow x) \wedge (\neg x \vee y)) \wedge \neg(\forall y.\,y \wedge x \vee \neg x \wedge \neg y))$
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- $\forall x.\,(\exists y.\,(\neg y \vee x) \wedge (\neg x \vee y)) \wedge (\exists z.\,(\neg z \vee \neg x) \wedge (x \vee z))$

$$\neg\forall x.\,\varphi \iff \exists x.\,\neg\varphi \qquad \neg\exists x.\,\varphi \iff \forall x.\,\neg\varphi$$

$$(\forall x.\,\varphi) \wedge \psi \iff \forall x.\,\varphi \wedge \psi \qquad \varphi \wedge \forall x.\,\psi \iff \forall x.\,\varphi \wedge \psi$$

$$(\exists x.\,\varphi) \wedge \psi \iff \exists x.\,\varphi \wedge \psi \qquad \varphi \wedge \exists x.\,\psi \iff \exists x.\,\varphi \wedge \psi$$

$$(\forall x.\,\varphi) \vee \psi \iff \forall x.\,\varphi \vee \psi \qquad \varphi \vee \forall x.\,\psi \iff \forall x.\,\varphi \vee \psi$$

$$(\exists x.\,\varphi) \vee \psi \iff \exists x.\,\varphi \vee \psi \qquad \varphi \vee \exists x.\,\psi \iff \exists x.\,\varphi \vee \psi$$

Example

- $\neg\exists x.\,\neg((\exists y.\,(y \rightarrow x) \wedge (\neg x \vee y)) \wedge \neg(\forall y.\,y \wedge x \vee \neg x \wedge \neg y))$
- $\neg\exists x.\,\neg((\exists y.\,(\neg y \vee x) \wedge (\neg x \vee y)) \wedge \neg(\forall y.\,y \wedge x \vee \neg x \wedge \neg y))$
- $\neg\exists x.\,\neg((\exists y.\,(\neg y \vee x) \wedge (\neg x \vee y)) \wedge \neg(\forall z.\,z \wedge x \vee \neg x \wedge \neg z))$
- $\forall x.\,(\exists y.\,(\neg y \vee x) \wedge (\neg x \vee y)) \wedge (\exists z.\,(\neg z \vee \neg x) \wedge (x \vee z))$
- $\forall x.\,\exists y\,z.\,(\neg y \vee x) \wedge (\neg x \vee y) \wedge (\neg z \vee \neg x) \wedge (x \vee z)$

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Outline

1. Nelson–Oppen Combination Method
2. Quantified Boolean Formulas
3. **PSPACE-Completeness of QBF**
4. Further Reading

Definitions

- PSPACE is class of decision problems that can be solved in polynomial space by deterministic Turing machine

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Theorem

- $\text{NP} \subseteq \text{PSPACE} = \text{NPSPACE}$ (open problem: $\text{NP} \subset \text{PSPACE}$)
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Theorem

- NP $\subseteq$ PSPACE $=$ NPSPACE (open problem: NP $\subset$ PSPACE)
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Some PSPACE-Complete Problems

- universality problem for regular expressions
- LTL model checking
- many games on $n \times n$ board (Othello, Go, Sokoban, Super Mario Bros.)

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TQBF Algorithm

input: closed QBF formula φ in prenex normal form

output: value of φ (true or false)

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Proof

recursive algorithm TQBF runs in linear space

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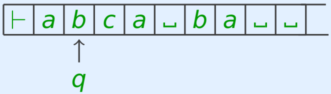

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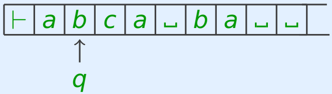

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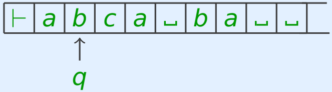

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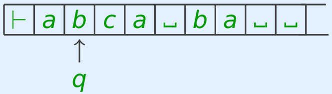

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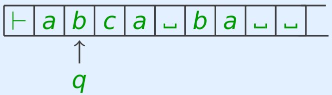

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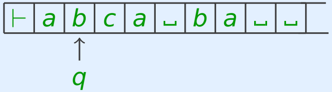

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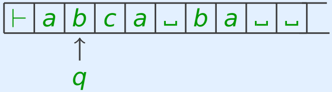

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$$\forall p \in Q : \text{if } \delta(p, \vdash) \text{ is defined then } \exists q \in Q\colon\ \delta(p, \vdash) = (q, \vdash, R)$$

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- $x \in \Sigma^*$ is accepted by M if $(s, \vdash x\llcorner\!\lrcorner^\omega, 0) \xrightarrow[M]{*} (t, y, n)$ for some y and n

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Definitions

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- continued on next slide . . .

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- size of $\varphi_M(x)$ is exponential in n ?!

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$$\varphi_{i+1}(c_1, c_2) = \exists c. \forall x\, y. \ulcorner x = c_1 \urcorner \wedge \ulcorner y = c \urcorner \vee \ulcorner x = c \urcorner \wedge \ulcorner y = c_2 \urcorner \to \varphi_i(x, y)$$

- $\varphi_M(x) = \varphi_m(a, b)$ with
  - start configuration a on input x
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  - bound m on required number of steps
- $m = \mathcal{O}(p(n))$
- size of $\varphi_M(x)$ is $\mathcal{O}(p^2(n))$

Outline

1. **Nelson–Oppen Combination Method**
2. **Quantified Boolean Formulas**
3. **PSPACE-Completeness of QBF**
4. **Further Reading**

Further Reading

we will not discuss methods for QBF-solving in this course;
interested in this topic? $\longrightarrow$ look at these chapters

- Hans Kleine Büning and Uwe Bubeck
  Theory of Quantified Boolean Formulas
  Chapter 29 of Handbook of Satisfiability (second edition)
  IOS Press, 2019
- Enrico Giunchiglia, Paolo Marin, and Massimo Narizzano
  Reasoning with Quantified Boolean Formulas
  Chapter 30 of Handbook of Satisfiability (second edition)
  IOS Press, 2019

Kröning and Strichmann

- Sections 9.1, 9.2
- Chapter 10

Bradley and Manna

- Section 10.1, 10.2, 10.3

Important Concepts

- arrangement
- convex theory
- purification
- stably infinite theory
- PSPACE
- QCNF
- quantified boolean formula
- TQBF