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SS 2024 lecture 11

Constraint Solving

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based on a previous course by Aart Middeldorp

Outline

Current State

- SMT solver for various theories
  - equality logic — equality graphs
  - EUF (equality with uninterpreted functions) — congruence closure
  - LRA (linear rational arithmetic) — simplex
  - LIA (linear integer arithmetic) — branch and bound
  - BV (bit-vector arithmetic) — flattening
- missing: combination of theories, e.g., EUF + LRA?

$$f(x+1) < f(y) \wedge f(f(x)) \neq f(x) - 1$$

- observation: all theories support equality
- upcoming: combination method which communicate via equalities
- requirement for presented combination algorithm: theories must be stably infinite

Definition

theory is stably infinite if every satisfiable quantifier-free formula has infinite model

Examples

- EUF is stably infinite
- LIA and LRA are stably infinite
- theory $T = (\Sigma, \mathcal{A})$ with $\Sigma = \{a, b, =\}$ and $\mathcal{A} = \{\forall x.\ x = a \vee x = b\}$ is not stably infinite
- BV is not stably infinite

Definition

(first-order) theory consists of

- signature Σ: set of function and predicate symbols
- axioms T: set of sentences in first-order logic in which only function and predicate symbols of Σ appear

Definition

theory combination $T_1 \oplus T_2$ of two theories

- T_1 over signature Σ_1
- T_2 over signature Σ_2

has signature $\Sigma_1 \cup \Sigma_2$ and axioms $T_1 \cup T_2$

Outline

Example

combination of linear arithmetic and uninterpreted functions:

$$x \geqslant y \;\wedge\; y - z \geqslant x \;\wedge\; \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \;\wedge\; z \geqslant 0$$

Assumptions

two stably infinite theories

- T_1 over signature Σ_1
- T_2 over signature Σ_2

such that

- $\Sigma_1 \cap \Sigma_2 = \{=\}$
- T_1-satisfiability of quantifier-free Σ_1-formulas is decidable
- T_2-satisfiability of quantifier-free Σ_2-formulas is decidable

Nelson–Oppen Method: Nondeterministic Version

input: quantifier-free conjunction φ in theory combination $T_1 \oplus T_2$

output: satisfiable or unsatisfiable

1. purification

$$\varphi \;\approx\; \varphi_1 \wedge \varphi_2 \quad \text{for } \Sigma_1\text{-formula } \varphi_1 \text{ and } \Sigma_2\text{-formula } \varphi_2$$

2. guess and check
   - V is set of shared variables in φ_1 and φ_2
   - guess equivalence relation E on V
   - arrangement $\alpha(V,E)$ is formula

$$\Big(\bigwedge_{x\,E\,y} x = y\Big) \;\wedge\; \Big(\bigwedge_{\neg(x\,E\,y)} x \neq y\Big)$$

   - if $\varphi_1 \wedge \alpha(V,E)$ is T_1-satisfiable and $\varphi_2 \wedge \alpha(V,E)$ is T_2-satisfiable then return satisfiable else return unsatisfiable

Example

formula φ in combination of LIA and EUF:

$$\underbrace{1 \leqslant x \ \wedge\ x \leqslant 2 \ \wedge\ y = 1 \ \wedge\ z = 2}_{\varphi_1} \wedge \underbrace{\mathsf{f}(x) \neq \mathsf{f}(y) \wedge \mathsf{f}(x) \neq \mathsf{f}(z)}_{\varphi_2}$$

- $V = \{x, y, z\}$
- 5 different equivalence relations E:
  1. $\{\{x, y, z\}\}$ $\quad\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
  2. $\{\{x, y\}, \{z\}\}$ $\quad\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
  3. $\{\{x, z\}, \{y\}\}$ $\quad\varphi_2 \wedge \alpha(V, E)$ is unsatisfiable
  4. $\{\{x\}, \{y, z\}\}$ $\quad\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
  5. $\{\{x\}, \{y\}, \{z\}\}$ $\quad\varphi_1 \wedge \alpha(V, E)$ is unsatisfiable
- φ is unsatisfiable

Outline

Definition

theory T is convex if for every quantifier-free conjunctive formula F and $n \geq 1$:

- If: $F \implies \bigvee_{i=1}^{n} x_i = y_i$
- Then: $F \implies x_i = y_i$ for some $1 \leqslant i \leqslant n$

Examples

- LIA is not convex

$$1 \leqslant x \leqslant 2 \ \wedge\ y = 1 \ \wedge\ z = 2 \quad \implies \quad x = y \ \vee\ x = z$$

- LRA is convex
- EUF is convex

Example

combination of LRA and EUF:

$$x \geqslant y \ \wedge\ y - z \geqslant x \ \wedge\ \mathsf{f}(\mathsf{f}(y) - \mathsf{f}(x)) \neq \mathsf{f}(z) \ \wedge\ z \geqslant 0$$

purification

$$\varphi_1: \quad x \geqslant y \ \wedge\ y - z \geqslant x \ \wedge\ w_1 = w_2 - w_3 \ \wedge\ z \geqslant 0$$
$$\varphi_2: \quad \mathsf{f}(w_1) \neq \mathsf{f}(z) \ \wedge\ w_2 = \mathsf{f}(y) \ \wedge\ w_3 = \mathsf{f}(x)$$

implied equalities between shared variables

$$\mathcal{E}: \ x = y \ \wedge\ w_2 = w_3 \ \wedge\ z = w_1$$

test satisfiability of $\varphi_2 \ \wedge\ \mathcal{E}$

$$\varphi_2 \ \wedge\ \mathcal{E} \quad \implies \quad \bot$$

unsatisfiable

Nelson–Oppen Method: Deterministic Version

input: quantifier-free conjunction φ in combination $T_1 \oplus T_2$ of convex theories

output: satisfiable or unsatisfiable

1. purification $\varphi \approx \varphi_1 \wedge \varphi_2$ for Σ_1-formula φ_1 and Σ_2-formula φ_2
2. V: set of shared variables in φ_1 and φ_2
   $\mathcal{E}$: already discovered equalities between variables in V
3. test satisfiability of $\varphi_1 \wedge \mathcal{E}$

   if $\varphi_1 \wedge \mathcal{E}$ is T_1-unsatisfiable then return unsatisfiable
   else add new implied equalities between variables in V to $\mathcal{E}$
4. test satisfiability of $\varphi_2 \wedge \mathcal{E}$

   if $\varphi_2 \wedge \mathcal{E}$ is T_2-unsatisfiable then return unsatisfiable
   else add new implied equalities between variables in V to $\mathcal{E}$
5. if $\mathcal{E}$ has been extended in steps 3 or 4 then goto step 2 else return satisfiable

Remark

Nelson–Oppen decision procedure can be extended to non-convex theories
$\Longrightarrow$ case-splitting for implied disjunction of equalities

Example

combination of LIA and EUF: $1 \leqslant x \wedge x \leqslant 2 \wedge \mathrm{f}(x) \neq \mathrm{f}(1) \wedge \mathrm{f}(x) \neq \mathrm{f}(2)$

purification

$$\varphi_1: \quad 1 \leqslant x \wedge x \leqslant 2 \wedge w_1 = 1 \wedge w_2 = 2$$
$$\varphi_2: \quad \mathrm{f}(x) \neq \mathrm{f}(w_1) \wedge \mathrm{f}(x) \neq \mathrm{f}(w_2)$$

implied equalities between shared variables (second case)

$$\mathcal{E}: \quad x = w_2$$

test satisfiability of $\varphi_2 \wedge \mathcal{E} \Longrightarrow \bot$

case split: $x = w_1$ or $x = w_2$ unsatisfiable

Requirement of Convex Theories for Deterministic Nelson–Oppen

- combine LIA (non-convex) + EUF (convex)
- problem with deterministic algorithm
  - $\underbrace{1 \leqslant x \wedge x \leqslant 2 \wedge y_1 = 1 \wedge y_2 = 2}_{\varphi_{LIA}} \wedge \underbrace{f(x) = a \wedge f(y_1) \neq a \wedge f(y_2) \neq a}_{\varphi_{EUF}}$
  - no implied equalities among shared variables $\{x, y_1, y_2\}$,
    hence deterministic algorithm wrongly reports satisfiability

Requirement of Stably Infinite Theories for Nelson–Oppen

- combine BV (not stably infinite) + EUF (stably infinite)
- problem with non-deterministic Nelson–Oppen algorithm
  - $\underbrace{x_4 \geqslant_u 0}_{\varphi_{BV}} \wedge \underbrace{\left(\bigwedge_{1 \leqslant i \leqslant 16} x_4 \neq a_i\right) \wedge \left(\bigwedge_{1 \leqslant i < j \leqslant 16} a_i \neq a_j\right)}_{\varphi_{EUF}}$
  - Nelson–Oppen reports satisfiability (only one trivial arrangement $\{x_4\}$,
    but the 4-bit vector x_4 cannot be different to 16 different constants $a_1, \dots, a_{16}$)

Outline

Current State

- SMT solving for various (combinations of) theories
- SMT: implicit existential quantification
- upcoming lectures: integrate quantifiers
- this lecture considers simplest case: Quantified Boolean Formulas

Definition (QBF Syntax)

$$\varphi \ ::= \ \bot \mid \top \mid x \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \forall x.\,\varphi \mid \exists x.\,\varphi$$

Definition (QBF Semantics)

extension of propositional logic case:

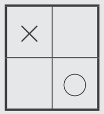
$$v(\forall x.\,\varphi) = \begin{cases} \mathsf{T} & \text{if } v[x/\mathsf{F}](\varphi) = \mathsf{T} \text{ and } v[x/\mathsf{T}](\varphi) = \mathsf{T} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

$$v(\exists x.\,\varphi) = \begin{cases} \mathsf{T} & \text{if } v[x/\mathsf{F}](\varphi) = \mathsf{T} \text{ or } v[x/\mathsf{T}](\varphi) = \mathsf{T} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

Definition

prenex normal form: $Q_1 x_1 \dots Q_n x_n.\,\varphi$ with $Q_i \in \{\forall, \exists\}$ for all $1 \leqslant i \leqslant n$ and φ quantifier-free

Example (2 × 2 Tic-Tac-Toe)



- two players (× and ○)
- first player (×) has winning strategy

$\exists$ × move (× wins or
$\forall$ ○ move (○ does not win and
$\exists$ × move × wins))

- expressible in QBF

Lemma

every quantified formula can be transformed into equivalent prenex normal form

Proof

1. eliminate all propositional connectives other than $\vee$, $\wedge$, $\neg$
2. rename all bound variables such that every quantifier binds unique variable
3. push propositional connectives through quantifiers:

$$\begin{aligned}
\neg\forall x.\,\varphi &\iff \exists x.\,\neg\varphi & \neg\exists x.\,\varphi &\iff \forall x.\,\neg\varphi \\
(\forall x.\,\varphi) \wedge \psi &\iff \forall x.\,\varphi \wedge \psi & \varphi \wedge \forall x.\,\psi &\iff \forall x.\,\varphi \wedge \psi \\
(\exists x.\,\varphi) \wedge \psi &\iff \exists x.\,\varphi \wedge \psi & \varphi \wedge \exists x.\,\psi &\iff \exists x.\,\varphi \wedge \psi \\
(\forall x.\,\varphi) \vee \psi &\iff \forall x.\,\varphi \vee \psi & \varphi \vee \forall x.\,\psi &\iff \forall x.\,\varphi \vee \psi \\
(\exists x.\,\varphi) \vee \psi &\iff \exists x.\,\varphi \vee \psi & \varphi \vee \exists x.\,\psi &\iff \exists x.\,\varphi \vee \psi
\end{aligned}$$

$$
\begin{array}{rcl@{\qquad}rcl}
\neg\forall x.\,\varphi & \iff & \exists x.\,\neg\varphi & \neg\exists x.\,\varphi & \iff & \forall x.\,\neg\varphi \\
(\forall x.\,\varphi)\wedge\psi & \iff & \forall x.\,\varphi\wedge\psi & \varphi\wedge\forall x.\,\psi & \iff & \forall x.\,\varphi\wedge\psi \\
(\exists x.\,\varphi)\wedge\psi & \iff & \exists x.\,\varphi\wedge\psi & \varphi\wedge\exists x.\,\psi & \iff & \exists x.\,\varphi\wedge\psi \\
(\forall x.\,\varphi)\vee\psi & \iff & \forall x.\,\varphi\vee\psi & \varphi\vee\forall x.\,\psi & \iff & \forall x.\,\varphi\vee\psi \\
(\exists x.\,\varphi)\vee\psi & \iff & \exists x.\,\varphi\vee\psi & \varphi\vee\exists x.\,\psi & \iff & \exists x.\,\varphi\vee\psi
\end{array}
$$

Example

- $\neg\exists x.\,\neg((\exists y.\,(y \rightarrow x)\wedge(\neg x\vee y))\wedge\neg(\forall y.\,y\wedge x\vee\neg x\wedge\neg y))$
- $\neg\exists x.\,\neg((\exists y.\,(\neg y\vee x)\wedge(\neg x\vee y))\wedge\neg(\forall y.\,y\wedge x\vee\neg x\wedge\neg y))$
- $\neg\exists x.\,\neg((\exists y.\,(\neg y\vee x)\wedge(\neg x\vee y))\wedge\neg(\forall z.\,z\wedge x\vee\neg x\wedge\neg z))$
- $\forall x.\,(\exists y.\,(\neg y\vee x)\wedge(\neg x\vee y))\wedge(\exists z.\,(\neg z\vee\neg x)\wedge(x\vee z))$
- $\forall x.\,\exists y\,z.\,(\neg y\vee x)\wedge(\neg x\vee y)\wedge(\neg z\vee\neg x)\wedge(x\vee z)$

Definition

closed QBF in prenex CNF: $Q_1 B_1 \dots Q_n B_n.\,\varphi$ with quantifier-free CNF φ QCNF

- $Q_i \in \{\forall, \exists\}$ for all $1 \leqslant i \leqslant n$ and $Q_i \neq Q_{i+1}$ for all $1 \leqslant i < n$
- B_i list of distinct variables with $B_i \cap B_j = \varnothing$ if $i \neq j$

QDIMACS Format, used in QBFLIB (QBF Satisfiability Library)

`p cnf 5 4`	5 variables 4 clauses
`e 1 3 4 0`	$\exists x_1\, x_3\, x_4$
`a 5 0`	$\forall x_5$
`e 2 0`	$\exists x_2$
`-1 2 0`	$(\neg x_1 \vee x_2)\wedge$
`3 5 -2 0`	$(x_3 \vee x_5 \vee \neg x_2)\wedge$
`4 -5 -2 0`	$(x_4 \vee \neg x_5 \vee \neg x_2)\wedge$
`-3 -4 0`	$(\neg x_3 \vee \neg x_4)$

Outline

Definitions

- PSPACE is class of decision problems that can be solved in polynomial space by deterministic Turing machine
- decision problem A is PSPACE-hard if every PSPACE problem B is poly-time reducible to A
- decision problem A is PSPACE-complete if it is PSPACE-hard and in PSPACE

Theorem

- NP $\subseteq$ PSPACE $=$ NPSPACE (open problem: NP $\subset$ PSPACE)
- QBF is PSPACE-complete

Some PSPACE-Complete Problems

- universality problem for regular expressions
- LTL model checking
- many games on $n \times n$ board (Othello, Go, Sokoban, Super Mario Bros.)

Lemma

QBF is in PSPACE

TQBF Algorithm

input: closed QBF formula φ in prenex normal form
output: value of φ (true or false)

1. if φ is quantifier-free then evaluate φ and return answer
2. if $\varphi = \exists x.\,\psi$ then return value of "TQBF($\psi[x/\bot]$) or TQBF($\psi[x/\top]$)"
3. if $\varphi = \forall x.\,\psi$ then return value of "TQBF($\psi[x/\bot]$) and TQBF($\psi[x/\top]$)"

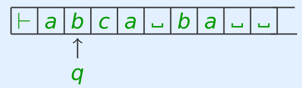
Proof

recursive algorithm TQBF runs in linear space

Definition

deterministic TM (DTM) is 8-tuple $M = (Q, \Sigma, \Gamma, \vdash, \text{␣}, \delta, s, t)$ with

1. Q: finite set of states
2. Σ: finite input alphabet
3. $\Gamma \supseteq \Sigma$: finite tape alphabet
4. $\vdash\, \in \Gamma - \Sigma$: left endmarker
5. $\text{␣} \in \Gamma - \Sigma$: blank symbol
6. $\delta\colon Q \times \Gamma \rightharpoonup Q \times \Gamma \times \{L, R\}$: (partial) transition function
7. $s \in Q$: start state
8. $t \in Q$: accept state



such that

$$\forall a \in \Gamma\colon\ \delta(t, a) \text{ is undefined}$$
$$\forall p \in Q : \text{if } \delta(p, \vdash) \text{ is defined then } \exists q \in Q\colon\ \delta(p, \vdash) = (q, \vdash, R)$$

Definitions

- configuration: element of $Q \times \{y\text{␣}^\omega \mid y \in \Gamma^*\} \times \mathbb{N}$
- start configuration on input $x \in \Sigma^*$: $(s, \vdash x\text{␣}^\omega, 0)$
- next configuration relation is binary relation $\xrightarrow[M]{1}$ defined as:

$$(p, z, n) \xrightarrow[M]{1} \begin{cases} (q, z', n-1) & \text{if } \delta(p, z_n) = (q, b, L) \\ (q, z', n+1) & \text{if } \delta(p, z_n) = (q, b, R) \end{cases}$$

  where z' is string obtained from z by substituting b for n-th symbol z_n of z

- $\xrightarrow[M]{n} = (\xrightarrow[M]{1})^n \quad \forall n \geqslant 0 \qquad \xrightarrow[M]{*} = \bigcup_{n \geqslant 0} \xrightarrow[M]{n}$
- $x \in \Sigma^*$ is accepted by M if $(s, \vdash x\text{␣}^\omega, 0) \xrightarrow[M]{*} (t, \vdash\text{␣}^\omega, 0)$
- $L(M)$ is set of strings accepted by M

Theorem

QBF is PSPACE-hard

Proof

- let A be arbitrary decision problem in PSPACE
- task: define polynomial-time reduction from A to QBF
- (language encoding of) A is accepted by DTM $M = (Q, \Sigma, \Gamma, \vdash, \text{␣}, \delta, s, t)$ that runs in polynomial space
- $\exists$ polynomial $p(n)$ such that M halts using at most $p(n)$ tape cells for any input x of length n
- given input x, we construct QBF formula $\varphi_M(x)$ such that

$$M \text{ accepts } x \iff \varphi_M(x) \text{ is true}$$

- continued on next slide ...

Proof (cont'd)

- reachable configurations of M on input x can be encoded using $\mathcal{O}(p(n))$ variables
- QBF formula $\varphi_m(c_1, c_2)$ encodes that c_2 can be reached from c_1 in at most 2^m steps:

$$\varphi_0(c_1, c_2) = \ulcorner c_1 = c_2 \urcorner \vee \ulcorner c_1 \xrightarrow[M]{1} c_2 \urcorner \qquad \text{(cf. NP-hardness proof of SAT)}$$

$$\varphi_{i+1}(c_1, c_2) = \exists c. \forall x\, y. \ulcorner x = c_1 \urcorner \wedge \ulcorner y = c \urcorner \vee \ulcorner x = c \urcorner \wedge \ulcorner y = c_2 \urcorner \to \varphi_i(x, y)$$

- $\varphi_M(x) = \varphi_m(a, b)$ with
  - start configuration a on input x
  - accept configuration b
  - bound m on required number of steps
- $m = \mathcal{O}(p(n))$
- size of $\varphi_M(x)$ is $\mathcal{O}(p^2(n))$

Outline

Further Reading

we will not discuss methods for QBF-solving in this course;
interested in this topic? ⟶ look at these chapters

- Hans Kleine Büning and Uwe Bubeck
  Theory of Quantified Boolean Formulas
  Chapter 29 of Handbook of Satisfiability (second edition)
  IOS Press, 2019
- Enrico Giunchiglia, Paolo Marin, and Massimo Narizzano
  Reasoning with Quantified Boolean Formulas
  Chapter 30 of Handbook of Satisfiability (second edition)
  IOS Press, 2019

Kröning and Strichmann

- Sections 9.1, 9.2
- Chapter 10

Bradley and Manna

- Section 10.1, 10.2, 10.3

Important Concepts

- arrangement
- convex theory
- purification
- stably infinite theory
- PSPACE
- QCNF
- quantified boolean formula
- TQBF