universität innsbruck

SS 2024 lecture 12

Constraint Solving

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based on a previous course by Aart Middeldorp

Outline

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Remark

since last lecture: consider formulas involving quantifiers

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  2. satisfiability of quantifier-free formulas is decidable

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Remarks

- first-order theory T is decidable if
  1. it admits quantifier elimination
  2. satisfiability of quantifier-free formulas is decidable
- quantifier elimination algorithm is needed only for formulas $\exists x.\ \varphi$ with quantifier-free φ
  - remove universal quantifiers: replace each $\forall x.\ \varphi$ by $\neg\exists x.\ \neg\varphi$
  - apply quantifier elimination algorithm starting with innermost quantifiers

Lemma

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- LRA formula $\exists x.\, 2x = y$ is equivalent to $\top$
- LIA formula $\exists x.\, 2x = y$ has no equivalent quantifier-free formula (over same signature)

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Upcoming

- LRA admits quantifier elimination (Ferrante and Rackoff’s method)
- augmented version of LIA admits quantifier elimination (Cooper’s method)

Lemma

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Remark

real arithmetic admits quantifier elimination (even non-linear)
(Tarski–Seidenberg, Collin's cylindrical algebraic decomposition algorithm)

Outline

1. Quantifier Elimination
2. **Ferrante and Rackoff's Method**
3. Cooper's Method
4. Further Reading

Ferrante and Rackoff's Method

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output: equivalent quantifier-free formula

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3. $\varphi_3(x)$ is result of replacing atoms in $\varphi_2(x)$ as follows:

$$t < cx \implies \begin{cases} \frac{t}{c} < x & \text{if } c > 0 \\ & \end{cases}$$

such that x does not appear in t

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Example

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- $\exists x.\ x < 3 \ \wedge \ x > \frac{13}{7}$

Ferrante and Rackoff's Method (cont'd)

4. $\varphi_3(x)$ is formula without negations and contains three types of atoms involving x:

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idea: finitely many cases for $\exists x.\varphi_3(x)$: x can be some element of S, between two elements of S, smaller than all elements of S or larger than all elements of S

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$$\varphi_4 = \varphi_{-\infty} \;\vee\; \varphi_{+\infty} \;\vee\; \bigvee_{s,t \in S} \varphi_3\left(\frac{s+t}{2}\right)$$

Example

- $\exists x.\, 3x + 1 < 10 \;\wedge\; 7x - 6 > 7$
- $\exists x.\, x < 3 \;\wedge\; x > \frac{13}{7}$
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- another quantifier elimination for y yields equivalent formula $\top$ via left infinite projection
- extract solution $y := -24$ and $x := \frac{\frac{y-1}{3} + \frac{7+y}{2}}{2} = -\frac{101}{12}$

Outline

Definition (Quantified Linear Integer Arithmetic)

- syntax:

$$\varphi \ ::= \ P \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\,\varphi \mid \forall x.\,\varphi$$

$$P \ ::= \ t = t \mid t < t$$

$$t \ ::= \ t + t \mid t - t \mid x \mid \cdots \mid -2 \mid -1 \mid 0 \mid 1 \mid 2 \mid \cdots$$

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- semantics: $\mathbb{Z}$ with standard interpretations

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- multiplication with constants is definable:

$$\begin{aligned} 2x &\quad \text{stands for} \quad x + x \\ -3x &\quad \text{stands for} \quad 0 - x - x - x \end{aligned}$$

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$$\exists x.\, 2x = y$$

Definition (Augmented Linear Integer Arithmetic)

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Definition (Augmented Linear Integer Arithmetic)

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\varphi &::= P \mid \neg\varphi \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid \exists x.\,\varphi \mid \forall x.\,\varphi \\
P &::= t=t \mid t<t \mid 1 \,|\, t \mid 2 \,|\, t \mid 3 \,|\, t \mid \cdots \\
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- quantifier elimination is possible

$$\exists x.\,2x=y \iff 2 \mid y$$

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Example

- $\exists x.\, \varphi \;=\; \exists x.\; -3x + 2y - 1 < y \;\wedge\; 2x - 6 < z \;\wedge\; 4 \mid 5x + 1 \;=\; \exists x.\, \varphi_1 \;=\; \exists x.\, \varphi_2$

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2. $\varphi_2(x)$ is result of replacing literals in $\varphi_1(x)$ as follows:

$$\begin{aligned} s = t \quad &\Longrightarrow \quad s < t + 1 \ \wedge \ t < s + 1 \\ \neg\,(s = t) \quad &\Longrightarrow \quad s < t \ \vee \ t < s \\ \neg\,(s < t) \quad &\Longrightarrow \quad t < s + 1 \end{aligned}$$

3. $\varphi_3(x)$ is result of collecting terms containing x such that literals have form

$$ax < t \qquad t < ax \qquad c \mid ax + t \qquad \neg\,(c \mid ax + t)$$

with $a > 0$

Example

- $\exists x.\, \varphi \;=\; \exists x.\, -3x + 2y - 1 < y \;\wedge\; 2x - 6 < z \;\wedge\; 4 \mid 5x + 1 \;=\; \exists x.\, \varphi_1 \;=\; \exists x.\, \varphi_2$
- $\exists x.\, \varphi_3 \;=\; \exists x.\, y - 1 < 3x \;\wedge\; 2x < z + 6 \;\wedge\; 4 \mid 5x + 1$

Cooper's Method (cont'd)

4. $\delta' = \text{lcm}\{\, a \mid a \text{ is coefficient of } x \text{ in } \varphi_3(x)\,\}$

Cooper's Method (cont'd)

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   multiply atoms in $\varphi_3(x)$ by constants such that δ' is coefficient of every x

Cooper's Method (cont'd)

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   multiply atoms in $\varphi_3(x)$ by constants such that δ' is coefficient of every x

   replace every occurrence of $\delta' x$ by fresh variable x'

Cooper's Method (cont'd)

4. $\delta' = \text{lcm}\,\{\, a \mid a \text{ is coefficient of } x \text{ in } \varphi_3(x)\,\}$

   multiply atoms in $\varphi_3(x)$ by constants such that δ' is coefficient of every x

   replace every occurrence of $\delta' x$ by fresh variable x'

   add conjunct $\delta' \mid x'$ to obtain $\varphi_4(x')$

Example

- $\exists x.\, \varphi \;=\; \exists x.\, -3x + 2y - 1 < y \;\wedge\; 2x - 6 < z \;\wedge\; 4 \mid 5x + 1 \;=\; \exists x.\, \varphi_1 \;=\; \exists x.\, \varphi_2$
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- $\delta' = \mathrm{lcm}\,\{3, 2, 5\} = 30$

Example

- $\exists x.\, \varphi \;=\; \exists x.\, -3x + 2y - 1 < y \;\wedge\; 2x - 6 < z \;\wedge\; 4 \mid 5x + 1 \;=\; \exists x.\, \varphi_1 \;=\; \exists x.\, \varphi_2$
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  $\exists x.\, 10y - 10 < 30x \;\wedge\; 30x < 15z + 90 \;\wedge\; 24 \mid 30x + 6$

Example

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Cooper's Method (cont'd)

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   replace every occurrence of $\delta' x$ by fresh variable x'

   add conjunct $\delta' \mid x'$ to obtain $\varphi_4(x')$

5. four types of literals in $\varphi_4(x')$:

   (A) $x' < t$ (B) $t < x'$ (C) $a \mid x' + t$ (D) $\neg\,(a \mid x' + t)$

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Example

- $\exists x.\,\varphi \;=\; \exists x.\;-3x + 2y - 1 < y \;\wedge\; 2x - 6 < z \;\wedge\; 4 \mid 5x + 1 \;=\; \exists x.\,\varphi_1 \;=\; \exists x.\,\varphi_2$
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- $\exists x'.\,\varphi_4 \;=\; \exists x'.\, 10y - 10 < x' \;\wedge\; x' < 15z + 90 \;\wedge\; 24 \mid x' + 6 \;\wedge\; 30 \mid x'$
- left infinite projection: $\varphi_{-\infty} = \bot \;\wedge\; \top \;\wedge\; 24 \mid x' + 6 \;\wedge\; 30 \mid x' \;\equiv\; \bot$

Cooper's Method (cont'd)

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Cooper's Method (cont'd)

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   B is set of terms t in (B) literals

$$\varphi_5 = \bigvee_{j=1}^{\delta} \varphi_{-\infty}(j) \ \vee \ \bigvee_{j=1}^{\delta} \bigvee_{t \in B} \varphi_4(t+j)$$

Example

- $\exists x.\, \varphi \;=\; \exists x.\, -3x + 2y - 1 < y \;\wedge\; 2x - 6 < z \;\wedge\; 4 \mid 5x + 1 \;=\; \exists x.\, \varphi_1 \;=\; \exists x.\, \varphi_2$
- $\exists x.\, \varphi_3 \;=\; \exists x.\, y - 1 < 3x \;\wedge\; 2x < z + 6 \;\wedge\; 4 \mid 5x + 1$
- $\delta' = \mathrm{lcm}\,\{3, 2, 5\} = 30$

  $\exists x.\, 10y - 10 < 30x \;\wedge\; 30x < 15z + 90 \;\wedge\; 24 \mid 30x + 6$
- $\exists x'.\, \varphi_4 \;=\; \exists x'.\, 10y - 10 < x' \;\wedge\; x' < 15z + 90 \;\wedge\; 24 \mid x' + 6 \;\wedge\; 30 \mid x'$
- left infinite projection: $\varphi_{-\infty} = \bot \wedge \top \wedge 24 \mid x' + 6 \wedge 30 \mid x' \equiv \bot$

  $\delta = \mathrm{lcm}\,\{24, 30\} = 120$

Example

- $\exists x.\, \varphi \;=\; \exists x.\; -3x + 2y - 1 < y \;\wedge\; 2x - 6 < z \;\wedge\; 4 \mid 5x + 1 \;=\; \exists x.\, \varphi_1 \;=\; \exists x.\, \varphi_2$
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  $\delta = \operatorname{lcm}\{24, 30\} = 120 \qquad B = \{10y - 10\}$

Example

- $\exists x.\, \varphi \;=\; \exists x.\, -3x + 2y - 1 < y \;\wedge\; 2x - 6 < z \;\wedge\; 4 \mid 5x + 1 \;=\; \exists x.\, \varphi_1 \;=\; \exists x.\, \varphi_2$
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  $\delta = \operatorname{lcm}\{24, 30\} = 120 \qquad B = \{10y - 10\}$

$$\varphi_5 \;=\; \bigvee_{j=1}^{120} \left[\begin{array}{l} 10y - 10 < 10y - 10 + j \;\wedge\; 10y - 10 + j < 15z + 90 \\ \wedge\; 24 \mid 10y - 10 + j + 6 \;\wedge\; 30 \mid 10y - 10 + j \end{array} \right]$$

Example

- $\exists x.\, \varphi \;=\; \exists x.\, -3x + 2y - 1 < y \;\wedge\; 2x - 6 < z \;\wedge\; 4 \mid 5x + 1 \;=\; \exists x.\, \varphi_1 \;=\; \exists x.\, \varphi_2$
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  $\delta = \text{lcm}\,\{24, 30\} = 120 \qquad B = \{10y - 10\}$

$$\varphi_5 \;=\; \bigvee_{j=1}^{120} \left[\begin{array}{l} 10y - 10 < 10y - 10 + j \;\wedge\; 10y - 10 + j < 15z + 90 \\ \wedge\; 24 \mid 10y - 10 + j + 6 \;\wedge\; 30 \mid 10y - 10 + j \end{array} \right]$$

$$\equiv\; \bigvee_{j=1}^{120} \left[\begin{array}{l} 0 < j \;\wedge\; 10y + j < 15z + 100 \\ \wedge\, 24 \mid 10y + j - 4 \;\wedge\; 30 \mid 10y + j - 10 \end{array} \right]$$

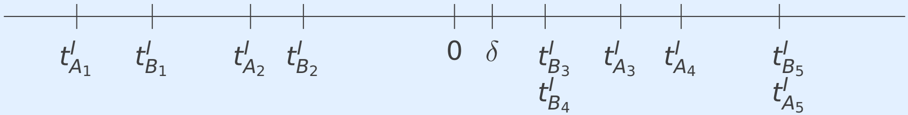
Outline

Elimination of Quantifier in Cooper's Method: From $\exists x.\varphi_4(x)$ to φ_5

- literals in NNF $\varphi_4(x)$: (A) $x < t$ (B) $t < x$ (C) $a \mid x + t$ (D) $\neg(a \mid x + t)$
- $\varphi_{-\infty}(x)$ is $\varphi_4(x)$ where (A) literals are replaced by $\top$ and (B) literals by $\bot$
- $\delta = \{a \mid a \text{ is constant of division predicate in } \varphi_4(x)\}$
- B is set of terms t in (B) literals
- $\varphi_5 = \bigvee_{j=1}^{\delta} \varphi_{-\infty}(j) \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} \varphi_4(t + j)$

Soundness Proof: $\exists x.\varphi_4(x)$ and φ_5 are Equivalent

- $\Longleftarrow$: if $\models_I \varphi_5$ then $\models_I \exists x.\varphi_4(x)$ for arbitrary environment I
- $\Longrightarrow$: if $\models_I \exists x.\varphi_4(x)$ then $\models_I \varphi_5$ for arbitrary environment I



Outline

5. four types of literals in $\varphi_4(x')$:

(A) $x' < t$ (B) $t < x'$ (C) $a \mid x' + t$ (D) $\neg(a \mid x' + t)$

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right infinite projection $\varphi_{+\infty}(x')$ is obtained from $\varphi_4(x')$ by replacing all (A) literals with $\bot$ and all (B) literals with $\top$

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A is set of terms t in (A) literals

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A is set of terms t in (A) literals

$$\varphi_5' = \bigvee_{j=1}^{\delta} \varphi_{+\infty}(-j) \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in A} \varphi_4(t - j)$$

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Left or Right Infinite Projection?

use right infinite projection if $|A| < |B|$ to reduce number of disjuncts

Eliminating Block of Quantifiers

$$\exists x_1 \ldots x_n.\, \varphi(x_1, \ldots, x_n)$$

Eliminating Block of Quantifiers

$$\exists x_1 \ldots x_n.\, \varphi(x_1,\ldots,x_n)$$

$$\equiv \exists x_1 \ldots x_{n-1}.\, \bigvee_{j=1}^{\delta} \varphi_{-\infty}(x_1,\ldots,x_{n-1},j) \;\vee\; \bigvee_{j=1}^{\delta} \bigvee_{t\in B} \varphi_4(x_1,\ldots,x_{n-1},t+j)$$

Eliminating Block of Quantifiers

$$
\begin{aligned}
&\exists x_1 \ \ldots\ x_n.\, \varphi(x_1, \ldots, x_n) \\
&\quad \equiv \exists x_1 \ \ldots\ x_{n-1}.\, \bigvee_{j=1}^{\delta} \varphi_{-\infty}(x_1, \ldots, x_{n-1}, j) \ \vee\ \bigvee_{j=1}^{\delta} \bigvee_{t \in B} \varphi_4(x_1, \ldots, x_{n-1}, t + j) \\
&\quad \equiv \bigvee_{j=1}^{\delta} \exists x_1 \ \ldots\ x_{n-1}.\, \varphi_{-\infty}(x_1, \ldots, x_{n-1}, j) \\
&\qquad \vee \bigvee_{j=1}^{\delta} \bigvee_{t \in B} \exists x_1 \ \ldots\ x_{n-1}.\, \varphi_4(x_1, \ldots, x_{n-1}, t + j)
\end{aligned}
$$

Eliminating Block of Quantifiers

$$\exists x_1 \ldots x_n.\, \varphi(x_1,\ldots,x_n)$$

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$$\equiv \bigvee_{j=1}^{\delta} \exists x_1 \ldots x_{n-1}.\, \varphi_{-\infty}(x_1,\ldots,x_{n-1},j)$$

$$\vee \bigvee_{j=1}^{\delta} \bigvee_{t\in B} \exists x_1 \ldots x_{n-1}.\, \varphi_4(x_1,\ldots,x_{n-1},t+j)$$

treat j as free variable and examine only $1+|B|$ formulas:

- $\exists x_1 \ldots x_{n-1}.\, \varphi_{-\infty}(x_1,\ldots,x_{n-1},j)$
- $\exists x_1 \ldots x_{n-1}.\, \varphi_4(x_1,\ldots,x_{n-1},t+j)$ for each $t \in B$

Outline

Bradley and Manna

- Sections 7.2 and 7.3

Bradley and Manna

- Sections 7.2 and 7.3

Further Reading

- Jeanne Ferrante and Charles Rackoff
  A Decision Procedure for the First Order Theory of Real Addition with Order
  SIAM Journal on Computing 4(1), pp. 69 – 76, 1975
- David C. Cooper
  Theorem Proving in Arithmetic without Multiplication
  Chapter 5 in Machine Intelligence 7, Edinburgh University Press, pp. 91 – 100, 1972

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Important Concepts

- augmented linear integer arithmetic
- Cooper's method
- divisibility constraint
- Ferrante and Rackoff's method
- left infinite projection
- quantifier elimination
- right infinite projection