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Interactive Theorem Proving using Isabelle/HOL

Session 8

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Outline

- Sets and Lists in Isabelle

- Practical Example: Binary Search Trees

Sets and Lists in Isabelle

Sets in Isabelle

- type ‘`'a set`’ for sets with elements of type `'a`

Set Basics

- `x ∈ A` – membership
- `A ∩ B` – intersection
- `A ∪ B` – union
- `-A` – complement
- `A - B` – difference
- `A ⊆ B` and `A ⊂ B` – subset
- `{}` – empty set
- `UNIV` – universal set (all elements of specific type)
- `{x}` – singleton set
- `insert x A` – insertion of single elements (`insert x A = {x} ∪ A`)
- ``f ` A`` – image of function with respect to set (“map `f` over elements of `A`”)

Further Operations on Sets

- `set` – convert list to set
- `Collect p` – convert predicate `p :: 'a ⇒ bool` to set of type `'a set`
- `finite A` – is set finite?
- `card A :: nat` – cardinality of set (note: `card A = 0` whenever `A` is infinite)
- `sum f A` – $\sum_{x \in A} f(x)$ (note: `sum f A = 0` whenever `A` is infinite)
- `prod f A` – similar to `sum`, just product
- `Ball A p` / `Bex A p` – do all / any elements of `A` satisfy predicate `p`?
- `Max A` and `Min A` – maximum and minimum of finite, non-empty set `A`
- `{x .. y}` – all elements between `x` and `y`

Syntax for Set Comprehension

- `{x . p x}` – same as `Collect p`
- `{t | x y. p x y}` – same as `{z. ∃ x y. t = z ∧ p x y}`
- example: `{ (x + 5, y) | x y. x < 7 ∧ odd y }`

Remarks on Finiteness and Cardinality

- properties like finiteness and cardinality do not work well in combination with set-comprehension or `Collect`
- in these cases it is often required to manually rewrite or estimate such sets by using images, products, intersections and unions
- since `card` returns a natural number, `card` does not work well with infinite sets; consequence: many lemmas on cardinalities have finiteness as assumption
- therefore, cardinality proofs are often accompanied by finiteness proofs

Demo – Example Proof

```
lemma "card { (x * 3, y) :: nat × bool | x y. x < 10 ∧ P y } ≤ 20"
```

Remarks on Sums and Products

- `sum f S = 0` and `prod f S = 1` whenever `S` is infinite
- infinite sums are available as limits, and will not be covered in this course
- there are several congruence lemmas on sums and products available, e.g., where the function `f` can be changed by a pointwise comparison
- there is ample special syntax for sums and products

Demo – Example Proof

```
lemma "sum (λ i. i) {..< (n :: nat)} ≤ n^2"
```

question: is lemma true, if `nat` is replaced by `int`?

Lists in Isabelle

- type '`'a list`' for lists with elements of type `'a`

List Basics – Selection of Functions

- `[]` or `Nil` and `#` or `Cons` – Nil and Cons
- `set` – conversion of list to set
- `length`, `take`, `drop`, `map`, `filter`, `concat`, `foldl`, `foldr` – as in Haskell
- `@` or `append` – append
- `hd` and `tl` – head and tail of list
- `xs ! n` – `n`-th element of `xs`
- `xs [ i := a ]` – list update, similar to function update `f (x := a)`

List Basics – Predicates

- `x ∈ set xs` – membership test via `set`
- `set xs ⊆ set ys` – sublist test via `set`
- `distinct`, `sorted`, ...

Syntax for Lists

- `[1, 3, x, 11, a + b]` – explicit finite list
- `[n ..< m]` – range, restricted to `nat list`
- `[n .. m]` – range, restricted to `int list`
- list comprehension is available, internally converted to `concat` and `map`; example
  - `[ (a, 2 * b) .  a <- [0 ..< n], even a, b <- [2 .. 5]]`

  -
    ```
    concat (map
        (λ a. if even a then map (λ b. (a, 2 * b)) [2..5] else [])
        [0..<n])
    ```

Reasoning on Lists and Sets

- automation works quite well for lists and sets
- still there are some lemmas which often have to be applied manually
  - all kinds of congruence rules or rules that work pointwise
    - `sum.cong` – `sum f A = sum g B` whenever `A = B` and `f x = g x` for all `x ∈ B`
    - `sum_mono` – `sum f A ≤ sum g A` whenever `f x ≤ g x` for all `x ∈ A`
    - `sum.neutral` – `sum f A = 0` whenever `f x = 0` for all `x ∈ A`
    - `nth_equalityI` – two lists are identical if they have the same length and are pointwise identical
  - `set_conv_nth` – definition of `set xs` via *n*-th elements
  - `split_list` – whenever `x ∈ set xs` then `xs = p @ x # s` for suitable `p` and `s`
- use find-theorems to gather existing results, e.g.,
  `find_theorems "sum _ (_ ∪ _) = _ + _"`

Practical Example: Binary Search Trees

Binary Search Tree

- binary tree: straight-forward datatype definition; tree is a leaf or a node storing an element with left- and right-subtree
- search tree: the tree is ordered, i.e., for each node with element x, left-subtree ℓ and right-subtree r, all elements in ℓ are strictly smaller than x and x is strictly smaller than all elements in r
- selected operations: insert, delete, and membership test
- optimizations are not included, e.g. balancing in splay-trees, AVL-trees, …

Demo and Exercise Session: Formalize Binary Search Trees