Interactive Theorem Proving

Lecture 1 (VU)

Cezary Kaliszyk

Administration

Teacher

- Cezary Kaliszyk
- Consultation hours: Wednesday midday, 3M12, on demand

Grading

- Exercises: Assignments and participation
- Presentations
- Closed book test (last week)

Practical Assignments

- Software: HOL Light, Coq, Set Theory
- Laptops convenient

Content

- Proof Assistants
- Lambda calculus
  - types, Church vs Curry, derivation formats, well-typedness, term finding
- Second-order typed lambda calculus
  - Π-types, second-order abstraction and application, $\lambda 2$
- Types dependent on types
  - Sorts, weakening, formation, properties
- Dependent types
  - λP, minimal logic, natural deduction again
- CoC
  - λ-cube, Girard's paradox, classical logic
- Definitions
  - terms, types, δ-conversion, $\rightarrow_\Delta$, axioms
- Sets and set theory
- Numbers and arithmetic
  - $\mathbb{N}$, bits, efficient computation, divisibility, proof irrelevance

Outline

Today

- Proof assistants
- Common uses
- Comparison with other tools

Course Prerequisites

- Propositional and predicate logic
- Functional programming

Automated Reasoning

- Computer used to reason in a logic
- Traditionally part of artificial intelligence
  - (not machine learning)
- Field of research since the fifties
- Applications: program verification, mathematical deduction, ...
- Theorem proving logics, precision, automation, ... very varied.

What is a Proof Assistant? (1/2)

A Proof Assistant is a

- a computer program
- to assist a mathematician
- in the production of a proof
- that is mechanically checked

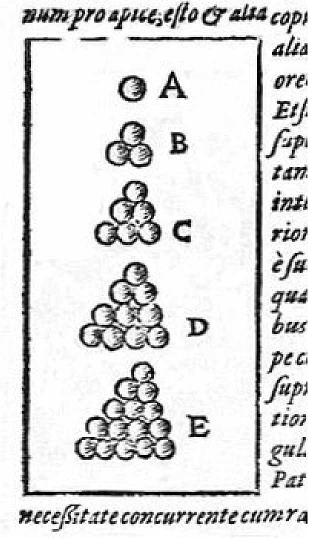
What does a Proof Assistant do?

- Keep track of theories, definitions, assumptions
- Interaction - proof editing
- Proof checking
- Automation - proof search

What does it implement? (And how?)

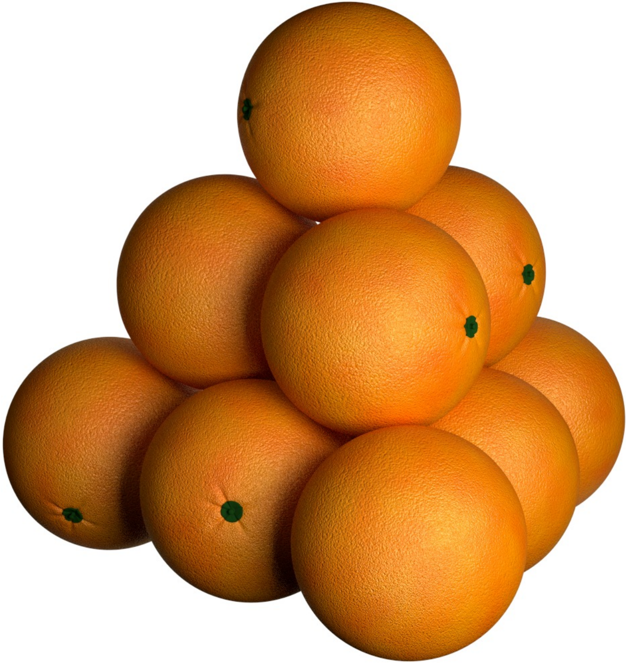
- a formal logical system intended as foundation for mathematics
- decision procedures

The Kepler Conjecture (year 1611)



The most compact way of stacking balls of the same size in space is a pyramid.

$$V = \frac{\pi}{\sqrt{18}} \approx 74\%$$



The Kepler Conjecture (year 1611)

Proved in 1998

- Tom Hales, 300 page proof using computer programs
- Submitted to the Annals of Mathematics

The Kepler Conjecture (year 1611)

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- 99% correct. . . but we cannot verify the programs

1039 equalities and inequalities

For example:

$$\frac{\begin{array}{c}-x_1x_3-x_2x_4+x_1x_5+x_3x_6-x_5x_6+\\ +x_2(-x_2+x_1+x_3-x_4+x_5+x_6)\end{array}}{\sqrt{4x_2\left(\begin{array}{c}x_2x_4(-x_2+x_1+x_3-x_4+x_5+x_6)+\\ +x_1x_5(x_2-x_1+x_3+x_4-x_5+x_6)+\\ +x_3x_6(x_2+x_1-x_3+x_4+x_5-x_6)-\\ -x_1x_3x_4-x_2x_3x_5-x_2x_1x_6-x_4x_5x_6\end{array}\right)}} < \tan\left(\frac{\pi}{2}-0.74\right)$$

The Kepler Conjecture (year 1611)

Solution? Formalized Proof!

- Formalize the proof using Proof Assistants
- Implement the computer code in the system
- Prove the code correct
- Run the programs inside the Proof Assistant

Flyspeck Project

- Completed 2017
- Many Proof Assistants and contributors

Intel Pentium P5 (1994)

FPU unit

- Division lookup table
- For certain inputs division result off

Replacement

- Few customers cared, still 450M$
- Birth of HOL Light
- Intel and AMD processors formally verified

```
theorem sqrt2_not_rational:
  "sqrt (real 2) ∉ ℚ"
proof

qed
```

```
theorem sqrt2_not_rational:
  "sqrt (real 2) ∉ ℚ"
proof
  assume "sqrt (real 2) ∈ ℚ"







































  thus False
qed
```

```
theorem sqrt2_not_rational:
  "sqrt (real 2) ∉ ℚ"
proof
  assume "sqrt (real 2) ∈ ℚ"
  then obtain m n :: nat where
    n_nonzero: "n ≠ 0" and sqrt_rat: "¦sqrt (real 2)¦ = real m / real n"
    and lowest_terms: "gcd m n = 1" ..
















  thus False
qed
```

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theorem sqrt2_not_rational:
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    and lowest terms: "gcd m n = 1" ..






          have eq: "m² = 2 * n²"
  hence "2 dvd m²" ..
                    have dvd m: "2 dvd m"




  hence "2 dvd n²" ..
                    have "2 dvd n"
             have "2 dvd gcd m n"

  thus False
qed
```

```
theorem sqrt2_not_rational:
  "sqrt (real 2) ∉ ℚ"
proof
  assume "sqrt (real 2) ∈ ℚ"
  then obtain m n :: nat where
    n_nonzero: "n ≠ 0" and sqrt_rat: "¦sqrt (real 2)¦ = real m / real n"
    and lowest_terms: "gcd m n = 1" ..
  from n_nonzero and sqrt_rat have "real m = ¦sqrt (real 2)¦ * real n" by simp
  then have "real (m²) = (sqrt (real 2))² * real (n²)"
    by (auto simp add: power2_eq_square)
  also have "(sqrt (real 2))² = real 2" by simp
  also have "... * real (m²) = real (2 * n²)" by simp
  finally have eq: "m² = 2 * n²" ..
  hence "2 dvd m²" ..
  with two_is_prime have dvd_m: "2 dvd m" by (rule prime_dvd_power_two)
  then obtain k where "m = 2 * k" ..
  with eq have "2 * n² = 2² * k²" by (auto simp add: power2_eq_square mult_ac)
  hence "n² = 2 * k²" by simp
  hence "2 dvd n²" ..
  with two_is_prime have "2 dvd n" by (rule prime_dvd_power_two)
  with dvd_m have "2 dvd gcd m n" by (rule gcd_greatest)
  with lowest_terms have "2 dvd 1" by simp
  thus False by arith
qed
```

```
:: The Irrationality of the Square Root of 2
theorem Th1:
 for p being Element of NAT st p is prime holds
 sqrt p is irrational
proof
 let p be Element of NAT ;
 assume A1: p is prime ;
 then A2: p > 1 by INT_2:def 4;
 assume sqrt p is rational ;
 then consider i being Integer, n being Element of NAT such that
 A3: n <> 0 and
 A4: sqrt p = i / n and
 A5: for i1 being Integer
 for n1 being Element of NAT st n1 <> 0 & sqrt p = i1 / n1 holds
 n <= n1 by RAT_1:9;
 A6: i = (sqrt p) * n by A3, A4, XCMPLX_1:87;
 sqrt p >= 0 by SQUARE_1:def 2;
 then reconsider m = i as Element of NAT by A6, INT_1:3;
 A7: m ^2 = ((sqrt p) ^2) * (n ^2) by A6
 .= p * (n ^2) by SQUARE_1:def 2 ;
 then p divides m ^2 by NAT_D:def 3;
 then p divides m by A1, NEWTON:80;
 then consider m1 being Nat such that
 A8: m = p * m1 by NAT_D:def 3;
 n ^2 = (p * (p * (m1 ^2))) / p by A2, A7, A8, XCMPLX_1:89
 .= p * (m1 ^2) by A2, XCMPLX_1:89 ;
 then p divides n ^2 by NAT_D:def 3;
 then p divides n by A1, NEWTON:80;
 then consider n1 being Nat such that
 A9: n = p * n1 by NAT_D:def 3;
 A10: m1 / n1 = sqrt p by A2, A4, A8, A9, XCMPLX_1:91;
 A11: n1 <> 0 by A3, A9;
 then p * n1 > 1 * n1 by A2, XREAL_1:98;
 hence contradiction by A5, A9, A11, A10;
end;
```

Proof Assistant (2/2)

- Keep track of theories, definitions, assumptions
  - set up a theory that describes mathematical concepts (or models a computer system)
  - express logical properties of the objects
- Interaction - proof editing
  - typically interactive
  - specified theory and proofs can be edited
  - provides information about required proof obligations
  - allows further refinement of the proof
  - often manually providing a direction in which to proceed.
- Automation - proof search
  - various strategies
  - decision procedures
- Proof checking
  - checking of complete proofs
  - sometimes providing certificates of correctness
- Why should we trust it?
  - small core

Can a Proof Assistant do all proofs?

Can a Proof Assistant do all proofs?

Decidability!

- Validity of formulas is undecidable
- (for non-trivial logical systems)

Automated Theorem Provers

- Specific domains
- Adjust your problem
- Answers: Valid (Theorem with proof)
- Or: Countersatisfiable (Possibly with counter-model)

Proof Assistants

- Generally applicable
- Direct modelling of problems

Other Tools

Computer Algebra

- Solving equations, simplifications, numerical approximations
- Maple, Mathematica, . . .

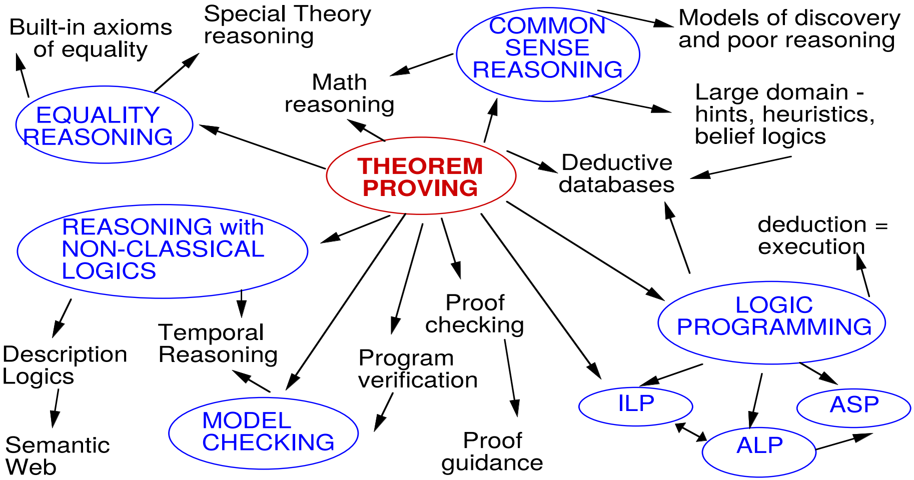
Model Checkers

- Space state abstraction
- Spin, Uppaal, . . .

ATPs

- Built in automation (model elimination, resolution)
- ACL2, Vampire, Eprover, SPASS, . . .

Spread of theorem proving (1/2)



[K. Broda]

Spread of theorem proving (2/2)

- Propositional logic
- SAT
- SMT
- SUMO
- First-order logic
- Resolution
- Orders
- Paramodulation
- Superposition
- Non-classical
- Description Logics
- Modal Logics
- ASP
- Higher-order logics
- Logical Frameworks
- Type Theory
- Set Theory
- Prolog (Lambda)
- ACL2

Users of Proof Assistants

Computer Science

- Modelling and specifying systems
- Proving properties of systems
- Proving software correct

Mathematics

- Defining concepts and theories
- Proving (mostly verifying) proofs
- (currently less common)

Theorems and programs that use ITP

Theorems

- Kepler Conjecture (2014)
- 4 color theorem
- Feit-Thomson theorem (2012)

Software

- Processors and Chips
- Security Protocols
- Project Cristal (Comp-Cert)
- L4-Verified
- Java Bytecode

History of Proof Assistants

λ-calculus (Church, 1940)

- Simple Type Theory
- Higher-Order Logic

Formulas as Types (Curry-Howard, de Bruijn)

- Proofs as Terms
- Reduce Proof Checking to Type Checking

Automath

- First implementation

LCF (Milner)

- ML programming language

Multitude of Proof Assistants

Characterized by various

- Foundations
- Interaction models
- Automation strategies
- Libraries
- Size of trusted core

Examples

- HOL (HOL4, HOL-Light, ProofPower, HOL0), Mizar (and variants), PVS, Coq, Otter/Ivy, Isabelle/Isar (HOL, ZF, CTT, . . .), Alfa/Agda, ACL2, IMPS, Metamath, Theorema, Lego, Nuprl, Ωmega, B method, Minlog

Coverage of Basic Mathematics

Freek Wiedijk's list of 100 theorems

Isabelle	89
HOL Light	87
Coq	79
Lean	76
MetaMath	74
Mizar	69
...	
any	99

`http://www.cs.ru.nl/~freek/100/`

Summary

This Lecture

- What is a Proof Assistant
- Common Uses
- Comparison with other tools
- Formal proof examples
- De Bruijn factor
- History
- Characteristics
- Coverage of Basic Mathematics

Next

- LCF and HOL Light
- Introduction to the λ-calculus

Homework / Work here

Have a look at an OCaml introduction and familiarize yourself with:

- Toplevel interaction (loading of files)
- Algebraic Datatypes
- Pattern Matching

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Tasks:

- How would you define the type of propositional logic terms?
- What about first-order logic?
- Can you define some basic operation like checking if a term is in CNF?