1 (a) *answer + explanation*

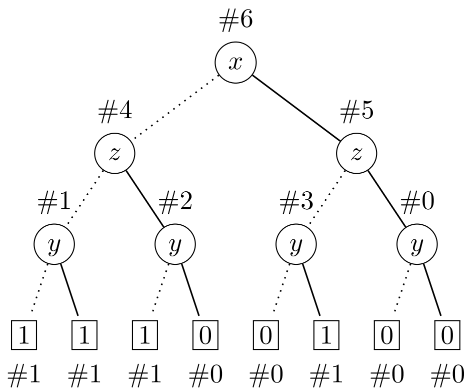
The BDD B_g is reduced because the rules C1, C2 and C3 are not applicable. It is not ordered since on different branches the variables are passed in different order, e.g. $[x, y, z]$ and $[x, z, y]$. Also, there are branches where x is visited twice.

(b) *answer + explanation*

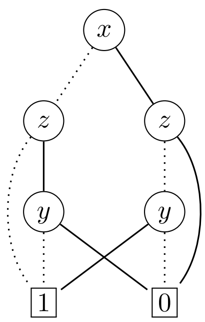
From the truth table

x	z	y	$g(x, y, z)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

we obtain the binary decision tree



Applying the reduce algorithm produces the desired reduced OBDD:



(c) *answer + explanation*

We have

$$\begin{aligned} f(x,y,z) &= (x \oplus y \oplus 1) \cdot (\overline{x} \oplus z \oplus \overline{x}z) \\ &= (x \oplus y \oplus 1) \cdot (x \oplus 1 \oplus z \oplus (x \oplus 1)z) \\ &= (x \oplus y \oplus 1) \cdot (x \oplus 1 \oplus z \oplus xz \oplus z) \\ &= (x \oplus y \oplus 1) \cdot (x \oplus 1 \oplus xz) \\ &= (x \oplus xy \oplus x) \oplus (x \oplus y \oplus 1) \oplus (xz \oplus xyz \oplus xz) \\ &= x \oplus y \oplus xy \oplus xyz \oplus 1 \end{aligned}$$

From the BDD in the solution of part (b) we obtain

$$\begin{aligned} g(x,y,z) &= \overline{x}(\overline{z} \oplus z\overline{y}) \oplus x\overline{z}y \\ &= (x \oplus 1)(z \oplus 1 \oplus z(y \oplus 1)) \oplus x(z \oplus 1)y \\ &= (x \oplus 1)(z \oplus 1 \oplus yz \oplus z) \oplus xyz \oplus xy \\ &= (x \oplus 1)(1 \oplus yz) \oplus xyz \oplus xy \\ &= x \oplus 1 \oplus xyz \oplus yz \oplus xyz \oplus xy \\ &= x \oplus xy \oplus yz \oplus 1 \end{aligned}$$

(d) *answer + explanation*

We have $f(0,0,0) = g(0,0,0) = f(1,1,1) = 1$ and $g(1,1,1) = 0$. Neither f nor g is monotone: $f(0,0,0) = 1 > 0 = f(0,1,0)$ and $g(0,0,0) = 1 > 0 = g(1,1,1)$. Moreover, $f(0,0,0) = f(1,1,1)$ and $g(1,0,0) = 0 = g(0,1,1)$, so f and g are not self-dual. The ANFs computed in part (c) are non-linear, so f and g are not affine. The following table summarizes our findings:

	f	g
$h(0,\cdots,0) \neq 0$	✓	✓
$h(1,\cdots,1) \neq 1$		✓
not monotone	✓	✓
not self-dual	✓	✓
not affine	✓	✓

(e) *answer + explanation*

We extend the table of part (d) with $\oplus$ and $+$:

	f	g	$\oplus$	$+$
$h(0, \cdots, 0) \neq 0$	✓	✓		
$h(1, \cdots, 1) \neq 1$		✓	✓	
not monotone	✓	✓	✓	
not self-dual	✓	✓	✓	✓
not affine	✓	✓		✓

It follows that a subset $S \subseteq \{f, g, \oplus, +\}$ is adequate if and only if $g \in S$ or both $f \in S$ and $\oplus \in S$.

2 (a) *answer + computation*

The following maximal derivation shows that the two terms are not unifiable:

$$\underline{f(h(z), g(x,x), z) \approx f(h(x), y, h(y))}$$

$$\Downarrow_{\mathsf{d}}$$

$$\underline{h(z) \approx h(x)},\ g(x,x) \approx y,\ z \approx h(y)$$

$$\Downarrow_{\mathsf{d}}$$

$$\underline{z \approx x},\ g(x,x) \approx y,\ z \approx h(y)$$

$$\mathsf{v} \Downarrow \{z \mapsto x\}$$

$$\underline{g(x,x) \approx y},\ x \approx h(y)$$

$$\mathsf{v} \Downarrow \{y \mapsto g(x,x)\}$$

$$\underline{x \approx h(g(x,x))}$$

$$\mathsf{failure} \Downarrow$$

$$\bot$$

(b) *answer + explanation*

We first rename the variables x and y in the second argument of the implication and then transform the resulting formula into an equivalent prenex normal form:

$$
\begin{aligned}
&\forall x\, \exists y\, (P(x) \to P(y)) \to \forall y\, \exists x\, Q(x, y)\\
&\quad \equiv\ \forall x\, \exists y\, (P(x) \to P(y)) \to \forall u\, \exists v\, Q(v, u)\\
&\quad \equiv\ \exists x\, \forall y\, ((P(x) \to P(y)) \to \forall u\, \exists v\, Q(v, u))\\
&\quad \equiv\ \exists x\, \forall y\, \forall u\, \exists v\, ((P(x) \to P(y)) \to Q(v, u))
\end{aligned}
$$

Next, we transform the quantifier-free part of the prenex normal form into CNF:

$$
\begin{aligned}
&\equiv\ \exists x\, \forall y\, \forall u\, \exists v\, (\neg(\neg P(x) \vee P(y)) \vee Q(v, u))\\
&\equiv\ \exists x\, \forall y\, \forall u\, \exists v\, ((P(x) \wedge \neg P(y)) \vee Q(v, u))\\
&\equiv\ \exists x\, \forall y\, \forall u\, \exists v\, ((P(x) \vee Q(v, u)) \wedge (\neg P(y) \vee Q(v, u)))
\end{aligned}
$$

We obtain an equisatisfiable Skolem normal form by replacing the existentially quantified variables x and v by the fresh Skolem constant c and the fresh Skolem function $g(y, u)$, respectively:

$$
\approx\ \forall y\, \forall u\, ((P(c) \vee Q(g(y, u), u)) \wedge (\neg P(y) \vee Q(g(y, u), u)))
$$

(c) *answer + explanation*

The clausal form is not satisfiable as seen by the following refutation:

1.	$\{Q(x),\ \neg P(f(x), f(x))\}$		
2.	$\{R(f(x), y),\ Q(x)\}$		
3.	$\{\neg Q(a)\}$		
4.	$\{R(x, f(y)),\ R(f(u), v)\}$		
5.	$\{P(x, y),\ \neg R(x, y)\}$		
6.	$\{\neg P(f(a), f(a))\}$	resolve 1, 3	$\{x \mapsto a\}$
7.	$\{\neg R(f(a), f(a))\}$	resolve 5, 6	$\{x \mapsto f(a),\ y \mapsto f(a)\}$
8.	$\{R(f(u), f(y))\}$	factor 4	$\{x \mapsto f(u),\ v \mapsto f(y)\}$
9.	$\square$	resolve 7, 8	$\{u \mapsto a,\ y \mapsto a\}$

3 (a) *answer*

The sequent $\neg(\neg q \wedge p) \vdash q \vee \neg p$ is valid:

1	$\neg(\neg q \wedge p)$	assumption
2	$q \vee \neg q$	LEM
3	q	assumption
4	$q \vee \neg p$	$\vee i_1$ 3
5	$\neg q$	assumption
6	p	assumption
7	$\neg q \wedge p$	$\wedge i$ 5, 6
8	$\bot$	$\neg e$ 1, 7
9	$\neg p$	$\neg i$ 6 − 8
10	$q \vee \neg p$	$\vee i_2$ 9
11	$q \vee \neg p$	$\vee e$ 2, 3 − 4, 5 − 10

(b) *answer*

The sequent $\forall x\, R(x) \vee \forall x \exists y\, S(x,y) \vdash \forall x\, \exists y\, (S(x,y) \vee R(x))$ is valid:

1		$\forall x\, R(x) \vee \forall x\, \forall y\, S(x,y)$	premise
2	x_0		
3		$\forall x\, R(x)$	assumption
4		$R(x_0)$	$\forall e$ 3
5		$S(x_0, y_0) \vee R(x_0)$	$\vee i_2$ 4
6		$\exists y\, (S(x_0, y) \vee R(x_0))$	$\exists i$ 5
7		$\forall x \exists y\, S(x,y)$	assumption
8		$\exists y\, S(x_0, y)$	$\forall e$ 7
9	y_0	$S(x_0, y_0)$	assumption
10		$S(x_0, y_0) \vee R(x_0)$	$\vee i_1$ 9
11		$\exists y\, (S(x_0, y) \vee R(x_0))$	$\exists i$ 10
12		$\exists y\, (S(x_0, y) \vee R(x_0))$	$\exists e$ 8, 9 − 11
13		$\exists y\, S(x_0, y) \vee R(x_0)$	$\vee e$ 1, 3 − 6, 7 − 11
14		$\forall x\, \exists y\, S(x,y) \vee R(x)$	$\forall i$ 2 − 13

(c) *answer*

The sequent $\forall x\, R(x) \wedge \forall x\, \exists y\, S(x,y) \vdash \exists y\, \forall x\, (S(x,y) \wedge R(x))$ is not valid. Consider the model $\mathcal{M}$ over the universe $A = \{a, b\}$ and the interpretations:

$$R^{\mathcal{M}} = \{a, b\} \qquad\qquad S^{\mathcal{M}} = \{(a,a), (b,b)\}$$

This model satisfies the premise, since both $\mathcal{M} \vDash \forall x\, R(x)$ and $\mathcal{M} \vDash \forall x\, \exists y\, S(x,y)$ hold. However the conclusion is not satisfied since for both choices of $y \in A$, there exists an $x \in A$ such that $(x,y) \notin S^{\mathcal{M}}$. Formally, $\mathcal{M} \nvDash_{\{y \mapsto a,\, x \mapsto b\}} S(x,y)$, hence $\mathcal{M} \nvDash_{\{y \mapsto a,\, x \mapsto b\}} S(x,y) \wedge R(x)$, and further $\mathcal{M} \nvDash_{\{y \mapsto a\}} \forall x\, (S(x,y) \wedge R(x))$. The same reasoning applies to the environment $\{y \mapsto b,\, x \mapsto a\}$, resulting in $\mathcal{M} \nvDash_{\{y \mapsto b\}} \forall x\, (S(x,y) \wedge R(x))$. Together these imply $\mathcal{M} \nvDash \exists y\, \forall x\, (S(x,y) \wedge R(x))$.

4 (a) *answer + explanation*

From the table

	p	q	$p \vee q$	$\neg q$	$\mathsf{EX}\,\neg q$	$q \rightarrow \mathsf{EX}\,\neg q$	$(p \vee q) \wedge (q \rightarrow \mathsf{EX}\,\neg q)$	φ
1	✓		✓	✓		✓	✓	✓
2		✓	✓					
3		✓	✓		✓	✓	✓	✓
4				✓	✓	✓		
5	✓		✓	✓	✓	✓	✓	✓

we conclude that the CTL formula $\varphi = \mathsf{EG}\,((p \vee q) \wedge (q \rightarrow \mathsf{EX}\,\neg q))$ holds in states 1, 3 and 5 of $\mathcal{M}$.

(b) *answer + explanation*

The requirement q restricts the possible initial states to 2 and 3. Because the next state may not satisfy q, this only leaves state 3. State 4 satisfies neither p nor q, so $p\,\mathsf{U}\,q$ is not satisfied, and the only possible successor is state 5. The path may loop in state 5, but to discharge $p\,\mathsf{U}\,q$, eventually the path needs to leave state 5. The only option to do so is via state 1, which satisfies p, and all its successors satisfy q, which then discharges $p\,\mathsf{U}\,q$. Thus, all paths satisfying the formula must have the prefix $3\,5^+\,1$, and all paths which start with this prefix satisfy the formula.

(c) *answer + explanation*

For instance,

$$\begin{aligned}\chi_1 &= p \wedge \neg\,\mathsf{EG}\,p\\ \chi_2 &= \mathsf{EG}\,q\\ \chi_3 &= q \wedge \mathsf{AF}\,\neg q\\ \chi_4 &= \mathsf{EG}(\neg p \wedge \neg q)\\ \chi_5 &= \mathsf{EG}\,p\end{aligned}$$

One easily checks that $\mathcal{M}, j \models \chi_i$ if and only if $j = i$:

	p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg\,\mathsf{EG}\,p$	$\mathsf{AF}\,\neg q$	χ_1	χ_2	χ_3	χ_4	χ_5
1	✓			✓		✓	✓	✓				
2		✓	✓			✓			✓			
3		✓	✓			✓	✓			✓		
4			✓	✓	✓	✓	✓				✓	
5	✓			✓			✓					✓

5	true	false	statement
	X		The sequent $p \to q \vdash \neg p \to \neg q$ is not valid.
		X	The function $f(x, y) = x \oplus y \oplus \overline{y}$ is monotone.
		X	Resolution is complete but not sound for predicate logic.
	X		Satisfaction of CTL formulas in finite models is decidable.
		X	In DPLL any backjump can always be simulated by a backtrack instead.
		X	It is not possible to verify if some comparator network is a sorting network.
		X	The CTL formula $p \wedge \mathsf{EX}\,\mathsf{EF}\,p$ is semantically equivalent to the LTL formula $\mathsf{F}\,p$.
	X		For every boolean function there exist at least two different reduced BDD representations. (*all answers received 2 points*)
	X		The Skolem normal form of a predicate logic formula cannot contain any existential quantifiers.
		X	The substitution $\{x \mapsto h(z),\ y \mapsto h(z)\}$ is a most general unifier of the terms $f(x, y, g(x))$ and $f(h(z), h(z), g(z))$.