1 (a) *answer + explanation*

Let the fresh variables $a_1, \ldots, a_6$ represent the non-atomic subformulas of φ:

$$a_1 = \varphi \qquad a_2 = p \wedge q \to r \vee \neg(q \wedge p) \qquad a_3 = p \wedge q$$
$$a_4 = r \vee \neg(q \wedge p) \qquad a_5 = \neg(q \wedge p) \qquad a_6 = q \wedge p$$

Using Tseitin's transformation we obtain

$$\begin{aligned}\varphi \approx\ & a_1 \wedge (a_1 \leftrightarrow \neg a_2)\\ & \wedge (a_2 \leftrightarrow (a_3 \to a_4))\\ & \wedge (a_3 \leftrightarrow p \wedge q)\\ & \wedge (a_4 \leftrightarrow r \vee a_5)\\ & \wedge (a_5 \leftrightarrow \neg a_6)\\ & \wedge (a_6 \leftrightarrow q \wedge p)\end{aligned}$$

which results in the equisatisfiable CNF

$$\begin{aligned}\phi \approx\ & a_1 \wedge (a_1 \vee a_2) \wedge (\neg a_1 \vee \neg a_2)\\ & \wedge (a_2 \vee a_3) \wedge (a_2 \vee \neg a_4) \wedge (\neg a_2 \vee \neg a_3 \vee a_4)\\ & \wedge (\neg a_3 \vee p) \wedge (\neg a_3 \vee q) \wedge (a_3 \vee \neg p \vee \neg q)\\ & \wedge (a_4 \vee \neg r) \wedge (a_4 \vee \neg a_5) \wedge (\neg a_4 \vee r \vee a_5)\\ & \wedge (a_5 \vee a_6) \wedge (\neg a_5 \vee \neg a_6)\\ & \wedge (\neg a_6 \vee q) \wedge (\neg a_6 \vee p) \wedge (a_6 \vee \neg q \vee \neg p)\end{aligned}$$

(b) *answer + explanation*

The following DPLL derivation shows that ψ is unsatisfiable:

$$
\begin{array}{lrcll}
 & & \| & \psi & \\
\Longrightarrow & \overset{d}{p} & \| & \psi & \text{(decide)} \\
\Longrightarrow & \overset{d}{p}\; t & \| & \psi & \text{(unit propagate)} \\
\Longrightarrow & \overset{d}{p}\; t\; r & \| & \psi & \text{(unit propagate)} \\
\Longrightarrow & \overset{d}{p}\; t\; r\; q & \| & \psi & \text{(unit propagate)} \\
\Longrightarrow & \neg p & \| & \psi & \text{(backtrack)} \\
\Longrightarrow & \neg p\; r & \| & \psi & \text{(unit propagate)} \\
\Longrightarrow & \neg p\; r\; q & \| & \psi & \text{(unit propagate)} \\
\Longrightarrow & \textsf{fail-state} & & & \text{(fail)}
\end{array}
$$

(c) *answer + explanation*

Recall the definition of semantic entailment: $\varphi_1, \ldots, \varphi_n \vDash \psi$ if $\overline{v}(\psi) = \mathsf{T}$ whenever $\overline{v}(\varphi_1) = \cdots = \overline{v}(\varphi_n) = \mathsf{T}$. It is easy to see that $\varphi_1, \ldots, \varphi_n \vDash \psi$ if and only if $\chi = \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \neg\psi$ is unsatisfiable: χ is unsatisfiable if and only if there is no valuation v under which all φ_i's evaluate to T but ψ evaluates to F. By definition, this is equivalent to the semantic entailment $\varphi_1, \ldots, \varphi_n \vDash \psi$. Furthermore, a maximal derivation in DPLL starting with χ leads to the fail-state if and only if χ is unsatisfiable. Therefore, in order to check $\varphi_1, \ldots, \varphi_n \vDash \psi$ by means of DPLL, we just have to construct χ and transform it into an (equisatisfiable) CNF χ'. Then, we can apply DPLL on χ' and the semantic entailment $\varphi_1, \ldots, \varphi_n \vDash \psi$ holds if and only if we can find a derivation in DPLL which reaches the fail-state.

For our concrete example, we obtain

$$\chi \;=\; (p \vee q \to r) \wedge (t \to p) \wedge t \wedge \neg r$$

and compute the following equivalent CNF:

$$\chi' \;=\; (\neg p \vee r) \wedge (\neg q \vee r) \wedge (\neg t \vee p) \wedge t \wedge \neg r$$

The following DPLL derivation shows that χ' is unsatisfiable:

$$
\begin{array}{lrcll}
 & & \| & \chi' & \\
\Longrightarrow & t & \| & \chi' & \text{(unit propagate)} \\
\Longrightarrow & t\; \neg r & \| & \chi' & \text{(unit propagate)} \\
\Longrightarrow & t\; \neg r\; p & \| & \chi' & \text{(unit propagate)} \\
\Longrightarrow & \textsf{fail-state} & & & \text{(fail)}
\end{array}
$$

Therefore, the semantic entailment $p \vee q \to r,\ t \to p,\ t \vDash r$ holds.

2 (a) *answer + computation*

The formula is not satisfiable. To show this we first transform it into CNF:

$$\varphi \;=\; (\neg p \vee \neg r) \wedge (p \vee \neg q \vee r) \wedge (r \vee q) \wedge (\neg p \vee \neg q) \wedge (\neg r \vee p)$$

Applying resolution leads to the following refutation:

1.	$\{\neg p, \neg r\}$	
2.	$\{p, \neg q, r\}$	
3.	$\{r, q\}$	
4.	$\{\neg p, \neg q\}$	
5.	$\{\neg r, p\}$	
6.	$\{\neg r\}$	resolve 1, 5, p
7.	$\{p, r\}$	resolve 2, 3, q
8.	$\{\neg p, r\}$	resolve 3, 4, q
9.	$\{r\}$	resolve 7, 8, p
10.	$\square$	resolve 6, 9, r

(b) *answer + explanation*

We start by renaming the variables, and transforming the formula to prenex normal form:

$$\begin{aligned}
\psi &= \exists x\,(\forall y\, P(g(x), y) \to \forall y\,(\exists z\, R(y, z) \to \exists x\, P(x, g(y)))) \\
&= \exists x_1\,(\forall y_1\, P(g(x_1), y_1) \to \forall y_2\,(\exists z\, R(y_2, z) \to \exists x_2\, P(x_2, g(y_2)))) \\
&= \exists x_1\,(\forall y_1\, P(g(x_1), y_1) \to \forall y_2\, \exists x_2\,(\exists z\, R(y_2, z) \to P(x_2, g(y_2)))) \\
&= \exists x_1\,(\forall y_1\, P(g(x_1), y_1) \to \forall y_2\, \exists x_2\, \forall z\,(R(y_2, z) \to P(x_2, g(y_2)))) \\
&= \exists x_1\, \exists y_1\,(P(g(x_1), y_1) \to \forall y_2\, \exists x_2\, \forall z\,(R(y_2, z) \to P(x_2, g(y_2)))) \\
&= \exists x_1\, \exists y_1\, \forall y_2\, \exists x_2\, \forall z\,(P(g(x_1), y_1) \to (R(y_2, z) \to P(x_2, g(y_2))))
\end{aligned}$$

We then transform the quantifier free part of the formula to CNF:

$$= \exists x_1\, \exists y_1\, \forall y_2\, \exists x_2\, \forall z\,(\neg P(g(x_1), y_1) \lor \neg R(y_2, z) \lor P(x_2, g(y_2)))$$

Finally we remove the existential quantifiers by replacing x_1 by a, y_1 by b, and x_2 by $f(y_2)$:

$$\approx \forall y_2\, \forall z\,(\neg P(g(a), b) \lor \neg R(y_2, z) \lor P(f(y_2), g(y_2)))$$

(c) *answer + explanation*

The clausal form is not satisfiable as seen by the refutation:

1.	$\{\neg Q(x),\ \neg P(y, f(x))\}$		
2.	$\{P(f(f(f(a))), f(a))\}$		
3.	$\{P(x, y),\ \neg P(f(y), f(x))\}$		
4.	$\{Q(f(a))\}$		
5.	$\{P(a, f(f(a)))\}$	resolve 2, 3	$\{x \mapsto a,\ y \mapsto f(f(a))\}$
6.	$\{\neg P(y, f(f(a)))\}$	resolve 1, 4	$\{x \mapsto f(a)\}$
7.	$\square$	resolve 5, 6	$\{y \mapsto a\}$

3 (a) *answer*

The sequent $\neg s \lor \neg t,\ \top \to s \land t \vdash \neg s \land \neg t$ is valid:

1	$\neg s \lor \neg t$	premise
2	$\top \to s \land t$	premise
3	$\top$	$\top$i
4	$s \land t$	$\to$e 2, 3
5	s	$\land$e$_1$ 4
6	t	$\land$e$_2$ 4
7	$\neg s$	assumption
8	$\bot$	$\neg$e 5, 7
9	$\neg t$	assumption
10	$\bot$	$\neg$e 6, 9
11	$\bot$	$\lor$e 1, 7−8, 9−10
12	$\neg s \land \neg t$	$\bot$e 11

(b) *answer*

The sequent $\forall x\, \exists y\, (P(x) \to Q(x,y)),\ \neg\, \exists x\, Q(a,x) \vdash \neg \forall x\, P(x)$ is valid:

1		$\forall x\, \exists y\, (P(x) \to Q(x,y))$	premise
2		$\neg \exists x\, Q(a,x)$	premise
3		$\forall x\, P(x)$	assumption
4		$P(a)$	$\forall$e 3
5		$\exists y\, (P(a) \to Q(a,y))$	$\forall$e 1
6	y_0	$P(a) \to Q(a,y_0)$	assumption
7		$Q(a,y_0)$	$\to$e 6, 4
8		$\exists x\, Q(a,x)$	$\exists$i 7
9		$\bot$	$\neg$e 8, 2
10		$\bot$	$\exists$e 5, 6−9
11		$\neg \forall x\, P(x)$	$\neg$i 3−10

(c) *answer*

The sequent $\forall x\, \exists y\, (P(x) \to Q(x,y)),\ \neg\forall x\, P(x) \vdash \neg\exists x\, Q(a,x)$ is not valid. Take the model $\mathcal{M}$ with the universe $A = \{0\}$ and the following interpretations:

$$P^{\mathcal{M}} = \varnothing \qquad\qquad Q^{\mathcal{M}} = \{(0,0)\} \qquad\qquad a^{\mathcal{M}} = 0$$

We have $\mathcal{M} \models \forall x\, \exists y\, (P(x) \to Q(x,y))$ since $P^{\mathcal{M}} = \varnothing$ makes $P(x)$ false and the implication true for any x. We also have $\mathcal{M} \models \neg\forall x\, P(x)$ since $0 \notin P^{\mathcal{M}}$. On the other hand, $\mathcal{M} \not\models \neg\exists x\, Q(a,x)$ because $\mathcal{M} \models \exists x\, Q(a,x)$ since $(0,0) \in Q^{\mathcal{M}}$.

4 (a) *answer + explanation*

From the table

	a	$\mathsf{AX}\,a$	b	$\mathsf{EX}\,b$	$\mathsf{E}[\mathsf{AX}\,a\,\mathsf{U}\,\mathsf{EX}\,b]$	φ
1	✓			✓	✓	
2			✓			✓
3				✓	✓	✓
4			✓	✓	✓	✓
5	✓			✓	✓	
6		✓			✓	✓

we conclude that the CTL formula $\varphi = \mathsf{AX}\,\mathsf{E}[\mathsf{AX}\,a\,\mathsf{U}\,\mathsf{EX}\,b]$ holds in states 2, 3, 4 and 6 of $\mathcal{M}$.

(b) *answer + explanation*

Consider the LTL formula $\psi = \mathsf{X}\,\mathsf{X}\,b$. As b does not hold in state 5, the path $(6\,1\,5)^\omega$ shows that $\mathcal{M}, 6 \vDash \psi$ does not hold. Since b holds in state 2 we conclude that $\mathcal{M}, 2 \vDash \neg\psi$ does not hold by considering the path $6\,1\,2\,3\,4^\omega$.

(c) *answer + explanation*

For instance,

$$
\begin{aligned}
\chi_1 &= a \wedge \mathsf{EX}\, a \\
\chi_2 &= b \wedge \neg\, \mathsf{EX}\, a \\
\chi_3 &= \neg(a \vee b) \wedge \mathsf{EX}\, b \\
\chi_4 &= b \wedge \mathsf{EX}\, a \\
\chi_5 &= a \wedge \neg\, \mathsf{EX}\, a \\
\chi_6 &= \neg(a \vee b) \wedge \neg\, \mathsf{EX}\, b
\end{aligned}
$$

One easily checks that $\mathcal{M}, j \models \chi_i$ if and only if $j = i$:

	a	b	$\neg(a \vee b)$	$\mathsf{EX}\, a$	$\mathsf{EX}\, b$	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
1	✓			✓	✓	✓					
2		✓					✓				
3			✓	✓	✓			✓			
4		✓		✓	✓				✓		
5	✓				✓					✓	
6			✓	✓							✓

There are many other solutions. For instance, $\mathsf{AX}\, a$ also works for χ_6.

5

true	false	statement
X		The function $p + \overline{q + \overline{p}}$ is affine.
X		If $\neg\varphi$ is valid then φ is unsatisfiable.
	X	In LTL, if $\pi \vDash p$ then also $\pi \vDash \mathsf{G}\, p\, \mathsf{U}\, \neg p$.
X		The sequent $\exists x\, \forall y\, Q(x, y) \vdash \forall x\, \exists y\, Q(y, x)$ is valid.
	X	The formula $(\forall x\, P(x)) \to P(x) \vee \exists z\, Q(z)$ is a sentence.
X		The CTL formula $\mathsf{EF}\, p \vee \mathsf{AF}\, \neg p$ is satisfied in all states of all models.
X		Any valid propositional formula can be proven using natural deduction.
	X	For a given formula in DNF, it is easier to check validity than satisfiability.
	X	If two terms are unifiable, there is a unique most general unifier which unifies them.
X		If a comparator network with n inputs sorts 2^n different bit-strings of length n, it is a sorting network.