Selected Solutions

[2] Let φ denote the given formula $a \leftrightarrow ((p \to q) \wedge (\neg p \to r))$. We have

$$\varphi \equiv (\neg a \vee \neg p \vee q) \wedge \boxed{(a \vee p \vee \neg r)} \wedge \boxed{(a \vee \neg p \vee \neg q)} \wedge (\neg a \vee p \vee r)$$

This equivalence can be verified by comparing truth tables:

a	p	q	r	φ	$(\neg a \vee \neg p \vee q)$	$(a \vee p \vee \neg r)$	$(a \vee \neg p \vee \neg q)$	$(\neg a \vee p \vee r)$
T	T	T	T	T	T	T	T	T
T	T	T	F	T	T	T	T	T
T	T	F	T	F	F	T	T	T
T	T	F	F	F	F	T	T	T
T	F	T	T	T	T	T	T	T
T	F	T	F	F	T	T	T	F
T	F	F	T	T	T	T	T	T
T	F	F	F	F	T	T	T	F
F	T	T	T	F	T	T	F	T
F	T	T	F	F	T	T	F	T
F	T	F	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	F	T	F	T	T
F	F	T	F	T	T	T	T	T
F	F	F	T	F	T	F	T	T
F	F	F	F	T	T	T	T	T

[5] [(a)] We use propositional atoms x_{ac} with $a \in \{1, \dots, 9\}$ denoting a state

1: Burgenland	4: Oberösterreich	7: Tirol
2: Kärnten	5: Salzburg	8: Vorarlberg
3: Niederösterreich	6: Steiermark	9: Wien

and $c \in \{1, 2, 3\}$ denoting a color. If $v(x_{ac}) = \mathsf{T}$ then state a will receive color c. There are two types of constraints.

(1) Every state receives a unique color:

$$(x_{a1} \vee x_{a2} \vee x_{a3}) \wedge (\neg x_{a1} \vee \neg x_{a2}) \wedge (\neg x_{a1} \vee \neg x_{a3}) \wedge (\neg x_{a2} \vee \neg x_{a3})$$

for all $a \in \{1, \dots, 9\}$.

(2) Neighbouring states must receive different colors:

$$(\neg x_{a1} \vee \neg x_{b1}) \wedge (\neg x_{a2} \vee \neg x_{b2}) \wedge (\neg x_{a3} \vee \neg x_{b3})$$

for all $(a, b) \in \{(1,3), (1,6), (2,5), (2,6), (2,7), (3,4), (3,6), (3,9), (4,5), (4,6), (5,6), (5,7), (7,8)\}$.

Taking the conjunction of all these constraints results in a CNF formula that is satisfiable if and only if the map of Austria is 3-colorable.

(b) We use the mapping

$$x_{ij} \mapsto (i-1) \times 3 + j$$

to associate a unique number in $\{1, \ldots, 27\}$ with every propositional atom x_{ij}. The CNF of part (a) then gives rise to the DIMACS format given here. Using Z3 on this input produces

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s SATISFIABLE
v 1 -2 -3 4 -5 -6 -7 -8 9 10 -11 -12 -13 -14 15 -16 17 -18 -19 20 -21 22 -23 -24 25
-26 -27 0
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which translates to the following coloring:

Burgenland $\mapsto$ 1	Oberösterreich $\mapsto$ 1	Tirol $\mapsto$ 2
Kärnten $\mapsto$ 1	Salzburg $\mapsto$ 3	Vorarlberg $\mapsto$ 1
Niederösterreich $\mapsto$ 3	Steiermark $\mapsto$ 2	Wien $\mapsto$ 1