universität innsbruck

Logik	SS 2024	LVA 703026 + 703027
Week 3		March 21, 2024

Selected Solutions

1 (b) We use 9 propositional atoms $x_1, \ldots, x_9$:

x_i is true/false $\iff$ number i is colored blue/red

For each solution of the equation $a + b = c$, two clauses are added:

$$(x_a \vee x_b \vee x_c) \qquad (\neg x_a \vee \neg x_b \vee \neg x_c)$$

It follows that the answer for $n = 9$ is yes if and only if the formula

$$\begin{aligned}\varphi = {} & (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_4 \vee x_6) \wedge (\neg x_2 \vee \neg x_4 \vee \neg x_6) \wedge \\ & (x_1 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4) \wedge (x_2 \vee x_5 \vee x_7) \wedge (\neg x_2 \vee \neg x_5 \vee \neg x_7) \wedge \\ & (x_1 \vee x_4 \vee x_5) \wedge (\neg x_1 \vee \neg x_4 \vee \neg x_5) \wedge (x_2 \vee x_6 \vee x_8) \wedge (\neg x_2 \vee \neg x_6 \vee \neg x_8) \wedge \\ & (x_1 \vee x_5 \vee x_6) \wedge (\neg x_1 \vee \neg x_5 \vee \neg x_6) \wedge (x_2 \vee x_7 \vee x_9) \wedge (\neg x_2 \vee \neg x_7 \vee \neg x_9) \wedge \\ & (x_1 \vee x_6 \vee x_7) \wedge (\neg x_1 \vee \neg x_6 \vee \neg x_7) \wedge (x_3 \vee x_4 \vee x_7) \wedge (\neg x_3 \vee \neg x_4 \vee \neg x_7) \wedge \\ & (x_1 \vee x_7 \vee x_8) \wedge (\neg x_1 \vee \neg x_7 \vee \neg x_8) \wedge (x_3 \vee x_5 \vee x_8) \wedge (\neg x_3 \vee \neg x_5 \vee \neg x_8) \wedge \\ & (x_1 \vee x_8 \vee x_9) \wedge (\neg x_1 \vee \neg x_8 \vee \neg x_9) \wedge (x_3 \vee x_6 \vee x_9) \wedge (\neg x_3 \vee \neg x_6 \vee \neg x_9) \wedge \\ & (x_2 \vee x_3 \vee x_5) \wedge (\neg x_2 \vee \neg x_3 \vee \neg x_5) \wedge (x_4 \vee x_5 \vee x_9) \wedge (\neg x_4 \vee \neg x_5 \vee \neg x_9)\end{aligned}$$

is satisfiable. The corresponding DIMACS encoding is

```
p cnf 9 32
1 2 3 0 -1 -2 -3 0
2 4 6 0 -2 -4 -6 0
1 3 4 0 -1 -3 -4 0
2 5 7 0 -2 -5 -7 0
1 4 5 0 -1 -4 -5 0
2 6 8 0 -2 -6 -8 0
1 5 6 0 -1 -5 -6 0
2 7 9 0 -2 -7 -9 0
1 6 7 0 -1 -6 -7 0
3 4 7 0 -3 -4 -7 0
1 7 8 0 -1 -7 -8 0
3 5 8 0 -3 -5 -8 0
1 8 9 0 -1 -8 -9 0
3 6 9 0 -3 -6 -9 0
2 3 5 0 -2 -3 -5 0
4 5 9 0 -4 -5 -9 0
```

Executing `z3 -dimacs color.txt`, with the file `color.txt` containing the above DIMACS encoding, yields the output

```
s UNSATISFIABLE
```

So the answer for $n = 9$ is no.

3 (b)

1	$(p \to q) \to q$	assumption
2	$q \to p$	assumption
3	$\neg p$	assumption
4	$\neg q$	MT 2, 3
5	$\neg(p \to q)$	MT 1, 4
6	p	assumption
7	$\bot$	$\neg$e 6, 3
8	q	$\bot$e 7
9	$p \to q$	$\to$i 6–8
10	$\bot$	$\neg$e 9, 5
11	p	PBC 3–10
12	$(q \to p) \to p$	$\to$i 2–11
13	$((p \to q) \to q) \to ((q \to p) \to p)$	$\to$i 1–12

(c)

1	$p \wedge q$	assumption
2	$\neg p \vee \neg q$	assumption
3	$\neg p$	assumption
4	p	$\wedge$e$_1$ 1
5	$\bot$	$\neg$e 4, 3
6	$\neg q$	assumption
7	q	$\wedge$e$_2$ 1
8	$\bot$	$\neg$e 7, 6
9	$\bot$	$\vee$e 2, 3–5, 6–8
10	$\neg(\neg p \vee \neg q)$	$\neg$i 2–9
11	$p \wedge q \to \neg(\neg p \vee \neg q)$	$\neg$i 1–10

4 We have one introduction rule and two elimination rules for $\leftrightarrow$:

$$\dfrac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots \\ \varphi\end{array}}{\varphi \leftrightarrow \psi}\ \leftrightarrow\text{i} \qquad \dfrac{\varphi \leftrightarrow \psi \qquad \varphi}{\psi}\ \leftrightarrow\text{e}_1 \qquad \dfrac{\varphi \leftrightarrow \psi \qquad \psi}{\varphi}\ \leftrightarrow\text{e}_2$$

By interpreting $\varphi \leftrightarrow \psi$ as $(\varphi \to \psi) \wedge (\psi \to \varphi)$, the above rules are derivable from the basic rules of natural deduction:

$\leftrightarrow$i:

φ	given
$\vdots$	
ψ	
$\varphi \to \psi$	$\to$i
ψ	given
$\vdots$	
φ	
$\psi \to \varphi$	$\to$i
$(\varphi \to \psi) \wedge (\psi \to \varphi)$	$\wedge$i

$\leftrightarrow$e$_1$:

$(\varphi \to \psi) \wedge (\psi \to \varphi)$	given
φ	given
$\varphi \to \psi$	$\wedge$e$_1$
ψ	$\to$e

$\leftrightarrow$e$_2$:

$(\varphi \to \psi) \wedge (\psi \to \varphi)$	given
ψ	given
$\psi \to \varphi$	$\wedge$e$_2$
φ	$\to$e