Logik	SS 2024	LVA 703026 + 703027
Week 5		April 18, 2024

Selected Solutions

1 (b) First we transform the negation of the given formula into CNF:

$$\begin{aligned}
\neg(((p \to (p \to p)) \to p) \to (\neg p \to \bot)) &\equiv \neg(\neg((p \to (p \to p)) \to p) \vee (\neg p \to \bot)) \\
&\equiv ((p \to (p \to p)) \to p) \wedge \neg(\neg p \to \bot) \\
&\equiv (\neg(p \to (p \to p)) \vee p) \wedge \neg(p \vee \bot) \\
&\equiv (\neg(\neg p \vee (p \to p)) \vee p) \wedge (\neg p \wedge \top) \\
&\equiv ((p \wedge \neg(p \to p)) \vee p) \wedge \neg p \\
&\equiv ((p \wedge (p \wedge \neg p)) \vee p) \wedge \neg p \\
&\equiv (p \vee p) \wedge ((p \wedge \neg p) \vee p) \wedge \neg p \\
&\equiv (p \vee p) \wedge (p \vee p) \wedge (\neg p \vee p) \wedge \neg p
\end{aligned}$$

The resulting clausal form $\{\{p\}, \{\neg p, p\}, \{\neg p\}\}$ is unsatisfiable:

1. $\{p\}$
2. $\{\neg p, p\}$
3. $\{\neg p\}$
4. $\Box$ resolve 1, 3, p

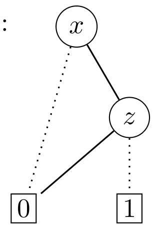
Hence the formula $(((p \to (p \to p)) \to p) \to (\neg p \to \bot)$ is valid.

2 The sequent $p \wedge q \to r \vdash (p \to r) \vee (q \to r)$ is valid:

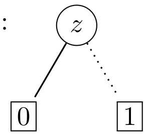
1	$p \wedge q \to r$	premise
2	$\neg((p \to r) \vee (q \to r))$	assumption
3	p	assumption
4	q	assumption
5	$p \wedge q$	$\wedge$i 3, 4
6	r	$\to$e 1, 5
7	$q \to r$	$\to$i 4–6
8	$(p \to r) \vee (q \to r)$	$\vee$i$_2$ 7
9	$\bot$	$\neg$e 2, 8
10	r	$\bot$e 9
11	$p \to r$	$\to$i 3–10
12	$(p \to r) \vee (q \to r)$	$\vee$i$_1$ 11
13	$\bot$	$\neg$e 2, 12
14	$(p \to r) \vee (q \to r)$	PBC 2–13

3 (c) Since $\forall y.h \equiv h[0/y] \cdot h[1/y]$, we start by computing OBDDs for $B_{h[0/y]}$ and $B_{h[1/y]}$:

$B_{h[0/y]}$:

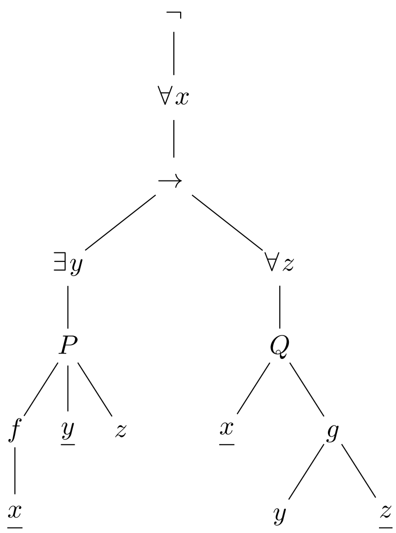


$B_{h[1/y]}$:



Since $\mathsf{apply}(\cdot, B_{h[0/y]}, B_{h[1/y]}) = B_{h[0/y]}$, the left OBDD represents $\forall y.h$.

4 (a)



There are 7 subformulas:

(1) $P(f(x), y, z)$,

(2) $\exists y\, P(f(x), y, z)$,

(3) $Q(x, g(y, z))$,

(4) $\forall z\, Q(x, g(y, z))$,

(5) $\exists y\, P(f(x), y, z) \to \forall z\, Q(x, g(y, z))$,

(6) $\forall x\, (\exists y\, P(f(x), y, z) \to \forall z\, Q(x, g(y, z)))$,

(7) $\neg\forall x\, (\exists y\, P(f(x), y, z) \to \forall z\, Q(x, g(y, z)))$.

(b) The underlined variable occurrences in the parse tree are bound, the others are free.

(c) i. We have

$$\begin{aligned}
\varphi[f(z)/x] &= \varphi \\
\varphi[f(z)/y] &= \neg\forall x\, (\exists y\, P(f(x), y, z) \to \forall z\, Q(x, g(f(z), z))) \\
\varphi[f(z)/z] &= \neg\forall x\, (\exists y\, P(f(x), y, f(z)) \to \forall z\, Q(x, g(y, z)))
\end{aligned}$$

The term $f(z)$ is free for x and z but not for y.

ii. We have

$$\begin{aligned}
\varphi[g(y, x)/x] &= \varphi \\
\varphi[g(y, x)/y] &= \neg\forall x\, (\exists y\, P(f(x), y, z) \to \forall z\, Q(x, g(g(y, x), z))) \\
\varphi[g(y, x)/z] &= \neg\forall x\, (\exists y\, P(f(x), y, g(y, x)) \to \forall z\, Q(x, g(y, x)))
\end{aligned}$$

The term $g(y, x)$ is free for x but not for y and z.

iii. We have

$$\begin{aligned}
\varphi[g(f(y), y)/x] &= \varphi \\
\varphi[g(f(y), y)/y] &= \neg\forall x\, (\exists y\, P(f(x), y, z) \to \forall z\, Q(x, g(g(f(y), y), z))) \\
\varphi[g(f(y), y)/z] &= \neg\forall x\, (\exists y\, P(f(x), y, g(f(y), y)) \to \forall z\, Q(x, g(y, z)))
\end{aligned}$$

The term $g(f(y), y)$ is free for x and y but not for z.

5 (a) First we show that adding a resolvent preserves satisfiability. Assume that S is satisfiable, i.e., there exists a valuation v such that $\bar{v}(S) = \mathsf{T}$. Consider clauses $C_1, C_2 \in S$ which clash on literal ℓ and let $C = \{(C_1 \setminus \{\ell\}) \cup (C_2 \setminus \{\ell^c\})\}$ and $S' = S \cup C$. If $\bar{v}(\ell) = \mathsf{F}$ then there exists a literal $\ell' \in C_1 \setminus \{\ell\}$ such that $\bar{v}(\ell') = \mathsf{T}$. Hence $\bar{v}(C) = \mathsf{T}$ and therefore S' is satisfiable. Otherwise, $\bar{v}(\ell) = \mathsf{T}$ and thus there exists a literal $\ell' \in C_2 \setminus \{\ell^c\}$ such that $\bar{v}(\ell') = \mathsf{T}$. Again, $\bar{v}(C) = \mathsf{T}$ and S' is satisfiable.

Now we can show the original claim. Let S be refutable, so resolution produces a clausal form S' with $\square \in S'$. For a proof by contradiction, assume S is satisfiable. By our previous reasoning, S' is also satisfiable. Since $\square \in S'$, we arrive at a contradiction. Hence S is unsatisfiable.