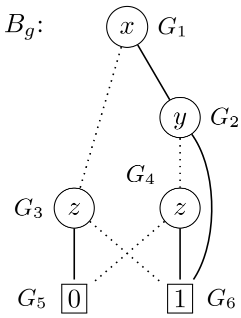
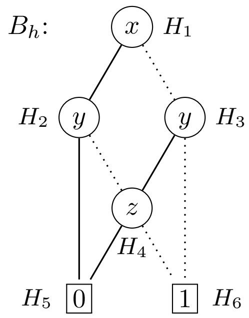
Logik | SS 2024 | LVA 703026 + 703027

Week 6 | April 25, 2024

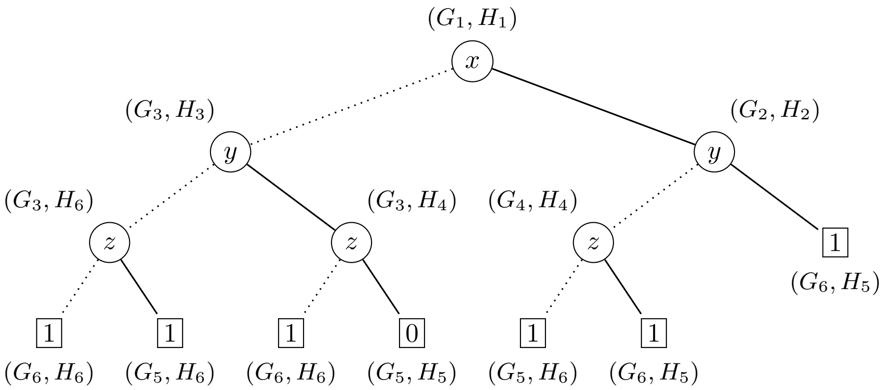
Selected Solutions

1 (b) For easy reference, we label the nodes of B_g and B_h:

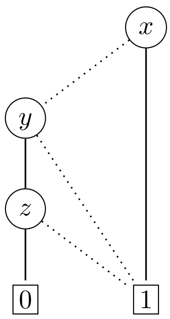




We obtain the intermediate OBDD



Applying the reduce algorithm produces the desired reduced OBDD B_{g+h}:



2 For instance, let $\mathcal{M}$ consist of the set $\{a, b\}$ with $f^{\mathcal{M}}(x, y) = a$ for all $x, y \in \{a, b\}$ and $P^{\mathcal{M}} = \{(a, a, a)\}$, and let the environment l satisfy $l(z) = a$. We have $\mathcal{M} \vDash_l \varphi$ because $f^{\mathcal{M}}(x, y) = f^{\mathcal{M}}(y, y) = a$ for all $x, y \in \{a, b\}$ and $(a, a, l(z)) \in P^{\mathcal{M}}$.

Let $\mathcal{M}'$ be the same model as $\mathcal{M}$ and let the environment l' satisfy $l'(z) = b$. We do not have $\mathcal{M}' \vDash_{l'} \varphi$ because $(a, a, l'(z)) = (a, a, b) \notin P^{\mathcal{M}'}$.

3 (b) The set $\{\forall x\,(\exists y\,P(x,y) \to \neg P(x,x)), \exists x\,\forall y\,P(y,x), \neg\forall x\,\neg P(x,x)\}$ is inconsistent. We show this by proving the validity of the sequent $\forall x\,(\exists y\,P(x,y) \to \neg P(x,x)),\ \exists x\,\forall y\,P(y,x) \vdash \forall x\,\neg P(x,x)$:

1		$\forall x\,(\exists y\,P(x,y) \to \neg P(x,x))$	premise
2		$\exists x\,\forall y\,P(y,x)$	premise
3	x_0	$\exists y\,P(x_0,y) \to \neg P(x_0,x_0)$	$\forall$e 1
4	x_1	$\forall y\,P(y,x_1)$	assumption
5		$P(x_0,x_1)$	$\forall$e 4
6		$\exists y\,P(x_0,y)$	$\exists$i 5
7		$\neg P(x_0,x_0)$	$\to$e 3, 6
8		$\neg P(x_0,x_0)$	$\exists$e 2, 4–7
9		$\forall x\,\neg P(x,x)$	$\forall$i 3–8

We obtain $\forall x\,(\exists y\,P(x,y) \to \neg P(x,x)),\ \exists x\,\forall y\,P(y,x) \vDash \forall x\,\neg P(x,x)$ by the soundness of natural deduction. Hence, any model $\mathcal{M}$ with $\mathcal{M} \vDash \forall x\,(\exists y\,P(x,y) \to \neg P(x,x))$ and $\mathcal{M} \vDash \exists x\,\forall y\,P(y,x)$ satisfies $\mathcal{M} \vDash \forall x\,\neg P(x,x)$ and therefore $\mathcal{M} \nvDash \neg\forall x\,\neg P(x,x)$.

4 (b) The sequent $\forall x\,P(f(x),x) \vdash \exists y\,\forall x\,P(y,x)$ is not valid. Take the model $\mathcal{M}$ with the universe $A = \{0,1\}$ and the interpretations

$$P^{\mathcal{M}} = \{(0,0),(1,1)\} \qquad\qquad f^{\mathcal{M}}(x) = x$$

We have $\mathcal{M} \vDash \forall x\,P(f(x),x)$, since both $(0,0) \in P^{\mathcal{M}}$ and $(1,1) \in P^{\mathcal{M}}$. However, for $y = 0$ we have $x = 1$ such that $(0,1) \notin P^{\mathcal{M}}$, and for $y = 1$ we have $x = 0$ with $(1,0) \notin P^{\mathcal{M}}$. Therefore $\forall x\,P(f(x),x) \nvDash \exists y\,\forall x\,P(x,y)$.

(c) The sequent $\exists x\,\exists y\,(P(x,y) \lor P(y,x)), \neg\exists x\,P(x,x) \vdash \exists x\,\exists y\,\neg(x = y)$ is valid:

1		$\exists x\,\exists y\,(P(x,y) \lor P(y,x))$	premise
2		$\neg\exists x\,P(x,x)$	premise
3	x_0	$\exists y\,(P(x_0,y) \lor P(y,x_0))$	assumption
4	y_0	$P(x_0,y_0) \lor P(y_0,x_0)$	assumption
5		$x_0 = y_0$	assumption
6		$P(y_0,y_0) \lor P(y_0,y_0)$	$=$e 5, 4
7		$P(y_0,y_0)$	assumption
8		$P(y_0,y_0)$	assumption
9		$P(y_0,y_0)$	$\lor$e 6, 7–7, 8–8
10		$\exists x\,P(x,x)$	$\exists$i 9
11		$\bot$	$\neg$e 10, 2
12		$\neg(x_0 = y_0)$	$\neg$i 5–11
13		$\exists y\,\neg(x_0 = y)$	$\exists$i 12
14		$\exists x\,\exists y\,\neg(x = y)$	$\exists$i 13
15		$\exists x\,\exists y\,\neg(x = y)$	$\exists$e 3, 4–14
16		$\exists x\,\exists y\,\neg(x = y)$	$\exists$e 1, 3–15