Selected Solutions

1 (b) The sequent $\vdash \forall x\, \forall y\, (P(x,y) \to \neg\forall z\, (P(x,z) \to \neg P(z,y))$ is not valid. For instance, consider the model $\mathcal{M}$ consisting of the set $\{a,b\}$ with interpretation $P^{\mathcal{M}} = \{(a,b)\}$ together with $l(x) = a$ and $l(y) = b$. We have $\mathcal{M} \vDash_l P(x,y)$ but not $\mathcal{M} \vDash_l \neg\forall z\, (P(x,z) \to \neg P(z,y))$. The latter can be seen as follows. If $l(z) = a$ then $\mathcal{M} \vDash_l P(x,z) \to \neg P(z,y)$ since $\mathcal{M} \nvDash_l P(x,z)$. If $l(z) = b$ then $\mathcal{M} \vDash_l \neg P(z,y)$ and thus also $\mathcal{M} \vDash_l P(x,z) \to \neg P(z,y)$. Hence $\mathcal{M} \nvDash_l \neg\forall z\, (P(x,z) \to \neg P(z,y))$.

3 (b) The following derivation shows that the given terms are not unifiable:

$$
\begin{array}{c}
\underline{P(x, f(y,a)) \approx P(y,x)} \\
\mathsf{d} \Downarrow \\
\underline{x \approx y},\ f(y,a) \approx x \\
\mathsf{v} \Downarrow \{x \mapsto y\} \\
f(y,a) \approx y \\
\Downarrow \\
\bot
\end{array}
$$

(c) The given atoms are unifiable:

$$
\begin{array}{c}
\underline{R(h(x,y,g(a,z))) \approx R(h(g(y,y),g(z,z),g(a,a)))} \\
\mathsf{d} \Downarrow \\
\underline{h(x,y,g(a,z)) \approx h(g(y,y),g(z,z),g(a,a))} \\
\mathsf{d} \Downarrow \\
x \approx g(y,y),\ y \approx g(z,z),\ \underline{g(a,z) \approx g(a,a)} \\
\mathsf{d} \Downarrow \\
x \approx g(y,y),\ y \approx g(z,z),\ \underline{a \approx a},\ z \approx a \\
\mathsf{t} \Downarrow \\
x \approx g(y,y),\ y \approx g(z,z),\ \underline{z \approx a} \\
\mathsf{v} \Downarrow \{z \mapsto a\} \\
x \approx g(y,y),\ \underline{y \approx g(a,a)} \\
\mathsf{v} \Downarrow \{y \mapsto g(a,a)\} \\
\underline{x \approx g(g(a,a),g(a,a))} \\
\mathsf{v} \Downarrow \{x \mapsto g(g(a,a),g(a,a))\} \\
\square
\end{array}
$$

The resulting mgu is $\{x \mapsto g(g(a,a),g(a,a)),\ y \mapsto g(a,a),\ z \mapsto a\}$.

4 We first rename the variable x in the second argument of the disjunction and then transform the resulting formula into an equivalent prenex normal form:

$$
\begin{aligned}
&\neg\big(\forall x\,\exists y\,P(f(x,y),y) \to \forall z\,Q(z)\big) \lor \forall x'\,R(x',x') \\
&\quad\equiv\ \neg\exists x\,\forall y\,\forall z\,\big(P(f(x,y),y) \to Q(z)\big) \lor \forall x'\,R(x',x') \\
&\quad\equiv\ \forall x\,\exists y\,\exists z\,\neg\big(P(f(x,y),y) \to Q(z)\big) \lor \forall x'\,R(x',x') \\
&\quad\equiv\ \forall x\,\exists y\,\exists z\,\forall x'\,\big(\neg\big(P(f(x,y),y) \to Q(z)\big) \lor R(x',x')\big)
\end{aligned}
$$

Next, we transform the quantifier-free part of the prenex normal form into CNF:

$$
\begin{aligned}
&\equiv\ \forall x\,\exists y\,\exists z\,\forall x'\,\big(\neg\big(\neg P(f(x,y),y) \lor Q(z)\big) \lor R(x',x')\big) \\
&\equiv\ \forall x\,\exists y\,\exists z\,\forall x'\,\big(\big(P(f(x,y),y) \land \neg Q(z)\big) \lor R(x',x')\big) \\
&\equiv\ \forall x\,\exists y\,\exists z\,\forall x'\,\big(\big(P(f(x,y),y) \lor R(x',x')\big) \land \big(\neg Q(z) \lor R(x',x')\big)\big)
\end{aligned}
$$

We obtain an equisatisfiable Skolem normal form by replacing the existentially quantified variables y and z by fresh Skolem functions $g(x)$ and $h(x)$:

$$
\approx\ \forall x\,\forall x'\,\big(\big(P(f(x,g(x)),g(x)) \lor R(x',x')\big) \land \big(\neg Q(h(x)) \lor R(x',x')\big)\big)
$$