Selected Solutions

1 (b) We start by renaming the variables with multiple bindings. In this case we replace the second x by v and the second y by w:

$$\varphi \equiv (\forall x\, \exists y\, P(f(x,y),x) \wedge \neg\forall z\, Q(z)) \rightarrow \forall v\, \neg\forall w\, R(w,v)$$

Now we apply quantifier equivalences to pull the quantifiers outside:

$$\begin{aligned}
&\equiv (\forall x\, \exists y\, P(f(x,y),x) \wedge \exists z\, \neg Q(z)) \rightarrow \forall v\, \exists w\, \neg R(w,v)\\
&\equiv \forall x\, \exists y\, (P(f(x,y),x) \wedge \exists z\, \neg Q(z)) \rightarrow \forall v\, \exists w\, \neg R(w,v)\\
&\equiv \forall x\, \exists y\, \exists z\, (P(f(x,y),x) \wedge \neg Q(z)) \rightarrow \forall v\, \exists w\, \neg R(w,v)\\
&\equiv \forall v\, \exists w\, (\forall x\, \exists y\, \exists z\, (P(f(x,y),x) \wedge \neg Q(z)) \rightarrow \neg R(w,v))\\
&\equiv \forall v\, \exists w\, \exists x\, \forall y\, \forall z\, ((P(f(x,y,x)) \wedge \neg Q(z)) \rightarrow \neg R(w,v))
\end{aligned}$$

At this point we have a prenex normal form. In the next step we transform the quantifier free part to CNF:

$$\begin{aligned}
&\equiv \forall v\, \exists w\, \exists x\, \forall y\, \forall z\, (\neg(P(f(x,y),x) \wedge \neg Q(z)) \vee \neg R(w,v))\\
&\equiv \forall v\, \exists w\, \exists x\, \forall y\, \forall z\, (\neg P(f(x,y),x) \vee Q(z) \vee \neg R(w,v))
\end{aligned}$$

In a final step we replace w and x by the unary Skolem function $g(v)$ and $h(v)$, respectively:

$$\approx \forall v\, \forall y\, \forall z\, (\neg P(f(h(v),y),h(v)) \vee Q(z) \vee \neg R(g(v),v))$$

2 (b) The clausal form is not satisfiable as seen by the following refutation:

1.	$\{P(x), P(f(a))\}$		
2.	$\{\neg P(y), \neg Q(f(x),y)\}$		
3.	$\{Q(f(a),a)\}$		
4.	$\{\neg Q(x,y), Q(f(y),x)\}$		
5.	$\{P(f(a))\}$	factor 1	$\{x \mapsto f(a)\}$
6.	$\{\neg Q(f(x), f(a))\}$	resolve 2, 5	$\{y \mapsto f(a)\}$
7.	$\{Q(f(a), f(a))\}$	resolve 3, 4	$\{x \mapsto f(a), y \mapsto a\}$
8.	$\square$	resolve 6, 7	$\{x \mapsto a\}$

3 We first calculate the truth tables of f_1, f_2 and f_3:

x_1	x_2	x_3	s	f_1	f_2	f_3
0	0	0	0	0	0	0
0	0	1	1	0	0	1
0	1	0	1	0	1	0
0	1	1	2	1	1	1
1	0	0	1	1	0	0
1	0	1	2	0	0	0
1	1	0	2	1	1	1
1	1	1	3	1	1	1

From the table we obtain

$$\begin{aligned}
f_1(x_1,x_2,x_3) &= \overline{x_1}x_2x_3 + x_1\overline{x_2}\,\overline{x_3} + x_1x_2 \;=\; \overline{x_1}x_2x_3 \oplus x_1\overline{x_2}\,\overline{x_3} \oplus x_1x_2 \\
&= (x_1 \oplus 1)x_2x_3 \oplus x_1(x_2 \oplus 1)(x_3 \oplus 1) \oplus x_1x_2 \\
&= x_1x_2x_3 \oplus x_2x_3 \oplus (x_1x_2 \oplus x_1)(x_3 \oplus 1) \oplus x_1x_2 \\
&= x_1x_2x_3 \oplus x_2x_3 \oplus x_1x_2x_3 \oplus x_1x_2 \oplus x_1x_3 \oplus x_1 \oplus x_1x_2 \\
&= x_1 \oplus x_1x_3 \oplus x_2x_3 \\
f_2(x_1,x_2,x_3) &= x_2 \\
f_3(x_1,x_2,x_3) &= \overline{x_1}x_3 + x_1x_2 \;=\; \overline{x_1}x_3 \oplus x_1x_2 \;=\; (x_1 \oplus 1)x_3 \oplus x_1x_2 \;=\; x_1x_2 \oplus x_1x_3 \oplus x_3
\end{aligned}$$

4 Using Shannon's expansion we obtain

$$\begin{aligned}
f(x,y,z) &= \overline{x}(\overline{y} \oplus y\overline{z}) \oplus x\overline{y}\,\overline{z} \\
&= (x \oplus 1)((y \oplus 1) \oplus y(z \oplus 1)) \oplus x(y \oplus 1)(z \oplus 1) \\
&= (x \oplus 1)(yz \oplus 1) \oplus x(yz \oplus y \oplus z \oplus 1) \\
&= xyz \oplus x \oplus yz \oplus 1 \oplus xyz \oplus xy \oplus xz \oplus x \\
&= xy \oplus xz \oplus yz \oplus 1
\end{aligned}$$