Selected Solutions

2 (a) Because $f(0,1,1) = 0 \oplus (1+1) = 1$ and $f(1,1,1) = 1 \oplus (1+1) = 0$, f is not monotone. The dual

$$\hat{f}(x,y,z) \;=\; \overline{f(\overline{x},\overline{y},\overline{z})} \;=\; \overline{\overline{x} \oplus (\overline{y} + \overline{z})} \;=\; \overline{\overline{x} \oplus \overline{yz}} \;=\; x \oplus yz \oplus 1$$

of f is different from f (e.g. $f(0,0,0) = 0$ and $\hat{f}(0,0,0) = 1$), so f is not self-dual. Finally, f is not affine since its algebraic normal form is $x \oplus y \oplus z \oplus yz$.

(b) No. Since $f(0,0,0) = 0$, it follows from Post's adequacy theorem that $\{f\}$ is not adequate. However, $\{f, \overline{}\}$ is adequate and hence $\overline{}$ cannot be expressed in terms of f.

Alternatively, one can prove by induction that any expression constructed from f and variables evaluates to 0 when all variables are set to 0. Since $\overline{0} = 1$, $\overline{}$ cannot be expressed.

3 Yes. By Post's adequacy theorem, $f(0,\dots,0) = 1$, $f(1,\dots,1) = 0$, and f is neither monotone nor self-dual nor affine. Let n be the arity of f. We prove that $\{\hat{f}\}$ is adequate by showing that $\hat{f}$ satisfies the five conditions of Post's adequacy theorem.

(1) We have $\hat{f}(0,\dots,0) = \overline{f(1,\dots,1)} = \overline{0} = 1$.

(2) We have $\hat{f}(1,\dots,1) = \overline{f(0,\dots,0)} = \overline{1} = 0$.

(3) Since f is not monotone,

$$f(b_1,\dots,b_{i-1},x,b_{i+1},\dots,b_n) \;=\; \overline{x}$$

for some i and $b_1,\dots,b_{i-1},b_{i+1},\dots,b_n \in \{0,1\}$. Hence also

$$f(b_1,\dots,b_{i-1},\overline{x},b_{i+1},\dots,b_n) \;=\; x$$

and thus

$$\hat{f}(\overline{b}_1,\dots,\overline{b}_{i-1},x,\overline{b}_{i+1},\dots,\overline{b}_n) \;=\; \overline{f(b_1,\dots,b_{i-1},\overline{x},b_{i+1},\dots,b_n)} \;=\; \overline{x}$$

It follows that $\hat{f}$ is not monotone.

Alternatively, the non-monotonicity of f is a direct consequence of (1) and (2).

(4) Since the dual $\hat{\hat{f}}$ of $\hat{f}$ is f, and $\hat{f} \neq f$ because f is not self-dual, $\hat{f}$ cannot be self-dual.

(5) If $\hat{f}$ is affine then there exist bits $c, c_1, \dots, c_n$ such that

$$\hat{f}(x_1,\dots,x_n) \;=\; c \oplus c_1x_1 \oplus \cdots \oplus c_nx_n$$

Hence

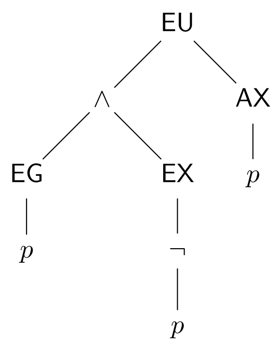
$$f(\overline{x}_1,\dots,\overline{x}_n) \;=\; \overline{c \oplus c_1x_1 \oplus \cdots \oplus c_nx_n} \;=\; 1 \oplus c \oplus c_1x_1 \oplus \cdots \oplus c_nx_n$$

and thus

$$f(x_1,\dots,x_n) \;=\; 1 \oplus c \oplus c_1\overline{x}_1 \oplus \cdots \oplus c_n\overline{x}_n \;=\; (1 \oplus c \oplus c_1 \oplus \cdots \oplus c_n) \oplus c_1x_1 \oplus \cdots \oplus c_nx_n$$

contradicting the fact that f is not affine. We conclude that $\hat{f}$ is not affine.

4 (a) From the parse tree of φ



we obtain 7 subformulas:

p $\mathsf{EG}\,p$ $\neg p$ $\mathsf{EX}\,\neg p$ $\mathsf{EG}\,p \wedge \mathsf{EX}\,\neg p$ $\mathsf{AX}\,p$ φ

(b) Applying the CTL model checking algorithm results in the table

	p	$\neg p$	$\mathsf{EG}\,p$	$\mathsf{EX}\,\neg p$	$\mathsf{EG}\,p \wedge \mathsf{EX}\,\neg p$	$\mathsf{AX}\,p$	φ
1	✓		✓	✓	✓		✓
2	✓		✓	✓	✓		✓
3		✓		✓			
4	✓		✓			✓	✓

Hence φ holds in states 1, 2 and 4.