



Logic

Diana Gründlinger

Aart Middeldorp

Fabian Mitterwallner

Alexander Montag

Johannes Niederhauser

Daniel Rainer

Outline

1. Introduction

Organisation

Motivation

Contents

2. Propositional Logic

3. Satisfiability and Validity

4. Intermezzo

5. Conjunctive Normal Forms

6. Further Reading

VO is **streamed** and **recorded**


ars.uibk.ac.at

with session ID **0992 9580** for anonymous questions



Important Information

- ▶ LVA 703026 (VO 3) + 703027 (PS 2)
- ▶ <http://cl-informatik.uibk.ac.at/teaching/ss24/lics>
- ▶ online registration for VO required until June 30
- ▶ OLAT links for VO and PS

Time and Place

VO	Monday	8:30 – 11:00	HSB 1	Aart		
TU	Wednesday	16:15 – 17:00	SR 13	Alexander		
PS	Thursday	12:00 – 13:30	HSB 8	Daniel	group 1	
	Thursday	8:30 – 10:00	HSB 4	Fabian	group 2	(in English)
	Thursday	8:30 – 10:00	SR 12	Johannes	group 3	(in English)
	Thursday	12:00 – 13:30	HS E	Diana	group 4	
	Thursday	13:45 – 15:15	HS 11	Diana	group 5	

PS group change requests until noon tomorrow using **SWAp** tool

Consultation Hours

Diana Gründlinger	3M03	Thursday 9:30–11:00
Aart Middeldorp	3M07	Wednesday 11:30–13:00
Fabian Mitterwallner	3M03	Thursday 10:30–12:00
Alexander Montag	ÖH Technik	Wednesday 14:00–15:00
Johannes Niederhauser	3M03	Friday 9:30–11:00
Daniel Rainer	3M03	Wednesday 11:30–13:00

Schedule

lecture 1	04.03 & 07.03	lecture 8	06.05 & 16.05
lecture 2	11.03 & 14.03	lecture 9	13.05 & 23.05
lecture 3	18.03 & 21.03	lecture 10	27.05 & 06.06
lecture 4	08.04 & 11.04	lecture 11	03.06 & 06.06
lecture 5	15.04 & 18.04	lecture 12	10.06 & 13.06
lecture 6	22.04 & 25.04	lecture 13	17.06 & 20.06
lecture 7	29.04 & 02.05	lecture 14	24.06 (first exam)

Announcements

- ▶ VO is streamed and recorded
- ▶ PS in presence, no PS on June 27

- ▶ first exam on June 24
- ▶ registration starts 5 weeks before exam and ends 2 weeks before exam
- ▶ late registration requests will be ignored
- ▶ de-registration is possible until 23:59 on June 20
- ▶ second exam on September 20
- ▶ third exam on February 26, 2025

score = $\min\left(\frac{50}{67}(E + P) + B, 100\right)$ E : points for solved **exercises** (at most 120)

B : points for **bonus exercises** (at most 20)

P : points for **presentations** of solutions (at most 14)

grade : $[0, 50) \rightarrow \mathbf{5}$ $[50, 63) \rightarrow \mathbf{4}$ $[63, 75) \rightarrow \mathbf{3}$ $[75, 88) \rightarrow \mathbf{2}$ $[88, 100] \rightarrow \mathbf{1}$

- ▶ homework exercises are given on course web site
- ▶ solved exercises must be marked in **OLAT**
- ▶ solutions must be uploaded (**PDF**) in OLAT; deadline: 7 am on Thursday
- ▶ 10 points per PS
- ▶ two presentations of solutions are mandatory
- ▶ 14 points for two presentations; additional presentations give bonus points
- ▶ attendance is compulsory; unexcused absence is allowed twice (resulting in 0 points)

Literature

Michael Huth and Mark Ryan

Logic in Computer Science (second edition)

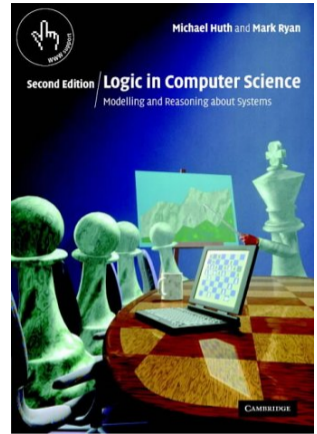
Cambridge University Press, 2004

in Semesterapparat

digital version

Online Material

- ▶ slides are available on Thursday before lecture on Monday
- ▶ solutions to selected exercises are available after they have been discussed in PS



evaluation SS 2023

- ▶ Es macht keinen sinn dass der Professor selbst viele Formeln nicht auswendig kann wir aber alle auswendig wissen müssen für den test. Das hat absolut nichts mit Verständnis zu tun. Es is unnötiges auswendig lernen was sich nichts bringt. In jeder anderen situation würde man z.b. die regeln der natural deduction oder Prädikaten logik nachlesen.

- ▶ The lecture was interesting, but the slides are very poorly structured or very often something is assumed but not explained why. It's a lot to memorize without the professor explaining/reciting it in a way that the student understands.

One of the worst lecturers, much is defined as it is, without explanation. And if students attend more different PS, then his subject is the most important thing to him, he doesn't take students into consideration.

- ▶ The examples given on the slides are, especially later during the semester, way to simple compared to whats wanted in the proseminar. More examples would be helpful, also during definitions as these can be rather dry otherwise.
- ▶ explained an extremely difficult topic very well. the logic course in of itself is complicated and extensive so i can't fault him for that. he knows his stuff for sure.
- ▶ This was the best VO in this semester. The slides were good, PS sheets helped a lot.

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ZEITTADEL der LFU Innsbruck

1669	Gründung der Universität Innsbruck aus dem seit 100 Jahren bestehenden Jesuitengymnasiums durch Leopold I.
1669/70	Aufnahme des Lehrbetriebs durch die Jesuiten. Erster Universitätskurs wird im Fach Logik abgehalten.
1677	Durch die Bestätigung der Errichtung durch Papst Innozenz XI. erlangt die LFU ihre volle Rechtsgültigkeit. Vor dem Hintergrund wissenschaftlich aufblühender protestantischer Hochschulen sollte Innsbruck das katholische Bollwerk zwischen Deutschland und Italien werden.
1781	Kaiser Joseph I. stuft die Universität Innsbruck zu einem Lyzeum zu Gunsten der Zentraluniversitäten Wien und Prag herab.
1792	Wiedereinrichtung durch Leopold II.
1809	Studentenkompanien beteiligen sich am Tiroler Freiheitskampf.
1810	Aufhebung durch die Bayern

WISSENSWERT

Mit Maria-Theresia kam die Bibliothek an die Universität:

Am 22. Mai 1745 genehmigte Maria Theresia die Errichtung einer Innsbrucker Bibliothek. Grundstein bildete die Büchersammlung der Tiroler Habsburger. Die Bibliothek war öffentlich zugänglich und die Benutzerordnung war streng: Es durfte immer nur ein Buch vor Ort gelesen werden und auf Bücherentwendungen stand die Exmatrikulation.

Innsbruck zieht Studierende an:
1684 meldet die Chronik des

Formal Logic at Department of Christian Philosophy

- ▶ During the past 30 years, there has been an extensive and growing interaction between logic and computer science.
- ▶ Concepts and methods of logic occupy a central place in computer science, insomuch that logic has been called **the calculus of computer science**.
- ▶ Logic has been much more effective in computer science than it has been in mathematics.

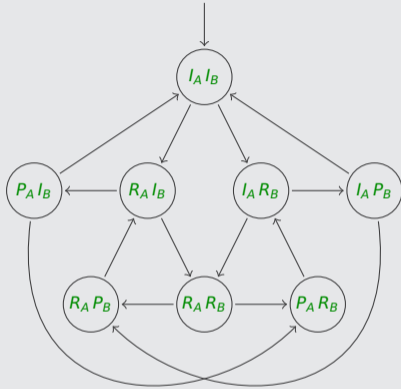
Phokion G. Kolaitis, Moshe Y. Vardi (2001)

Example (数独 Sudoku)

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

propositional logic is very useful to quickly develop efficient solver for Sudoku and all kinds of other tasks

Example (Printer Manager)



two users A and B

I_i user i is idle

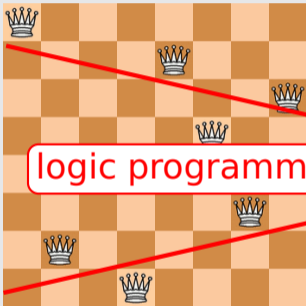
R_i print request by user i

P_i printing document for user i

some questions

- ▶ is every P_i preceded by R_i ?
- ▶ is every R_i eventually followed by P_i ?

Example (Eight Queens Puzzle)

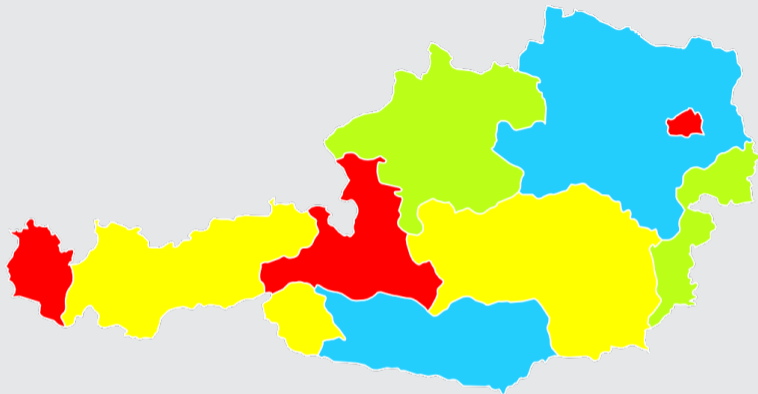


logic programming is sometimes taught in elective module

Prolog code

```
:- use_module(library(clpfd)).  
nqueens(N,Qs) :-  
    length(Qs,N), Qs ins 1 .. N,  
    all_different(Qs),  
    constraint_queens(Qs), label(Qs).  
constraint_queens([])  
    !, constraint_queens(Qs).  
constraint_queens(Qs).  
noattack(_,[],_).  
noattack(X,[Q|Qs],N) :-  
    X #\= Q+N, X #\= Q-N, M is N+1,  
    noattack(X,Qs,M).  
  
?- nqueens(8,Xs).
```


Example (Coloring Austria)



question: do three colors suffice to color Austria ?

Example (Programming Task)

function that computes $\sum_{i=1}^n i$ (exercise in C programming course at Nagoya University)

some answers:

```
int sum(int x) {
    int i, j, z;
    z = 0;
    for (i = 0; i <= x; i++)
        for (j = 0; j < i; j++)
            z++;
    return z;
}
```

```
int sum(int n) {
    if (n <= 0) {
        return 0;
    } else {
        return (n*(n+1)/2);
    }
}
```

question: are these programs correct ?

Greek Alphabet

alpha	α	A	eta	η	H	nu	ν	N	tau	τ	T
beta	β	B	theta	$\theta \vartheta$	Θ	xi	ξ	Ξ	upsilon	υ	Υ
gamma	γ	Γ	iota	ι	I	omicron	\omicron	O	phi	$\phi \varphi$	Φ
delta	δ	Δ	kappa	κ	K	pi	π	Π	chi	χ	X
epsilon	$\epsilon \varepsilon$	E	lambda	λ	Λ	rho	ρ	P	psi	ψ	Ψ
zeta	ζ	Z	mu	μ	M	sigma	$\sigma \varsigma$	Σ	omega	ω	Ω

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Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, **conjunctive normal forms**, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, **semantics**, sorting networks, soundness and completeness, **syntax**, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching-time temporal logic, CTL*, fairness, linear-time temporal logic, model checking algorithms, symbolic model checking

Outline

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Syntax

Semantics

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Definition

propositional **formulas** are built from

- ▶ **atoms** p, q, r, p_1, p_2, \dots
- ▶ **bottom** \perp
- ▶ **top** \top
- ▶ **negation** \neg $\neg p$ "not p "
- ▶ **conjunction** \wedge $p \wedge q$ " p and q "
- ▶ **disjunction** \vee $p \vee q$ " p or q "
- ▶ **implication** \rightarrow $p \rightarrow q$ "if p then q "

according to following **Backus–Naur Form**:

$$\varphi ::= p \mid \perp \mid \top \mid (\neg \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi)$$

Notational Conventions

- ▶ **binding precedence** $\neg > \wedge, \vee > \rightarrow$
- ▶ omit outer parentheses
- ▶ $\rightarrow, \wedge, \vee$ are **right-associative**: $p \rightarrow q \rightarrow r$ denotes $p \rightarrow (q \rightarrow r)$

Example

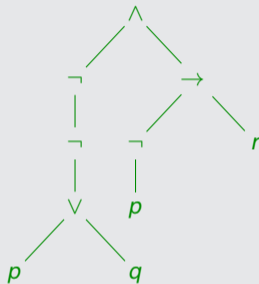
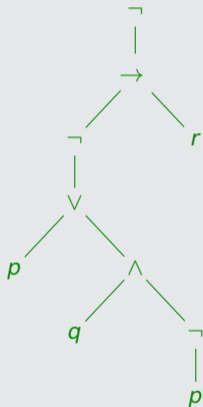
formula

$$\neg(\neg(p \vee (q \wedge \neg p)) \rightarrow r)$$

$$\neg\neg(p \vee q) \wedge (\neg p \rightarrow r)$$

$$\neg\neg p \vee q \wedge \neg p \rightarrow r$$

parse tree



?

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Definitions (Boole 1854)

▶ **valuation** (truth assignment) is mapping $v: \{p \mid p \text{ is atom}\} \rightarrow \{T, F\}$

▶ \bar{v} is extension of v to formulas:

truth values

$$\bar{v}(p) = v(p)$$

$$\bar{v}(\varphi \vee \psi) = \begin{cases} F & \text{if } \bar{v}(\varphi) = \bar{v}(\psi) = F \\ T & \text{otherwise} \end{cases}$$

$$\bar{v}(\perp) = F$$

$$\bar{v}(T) = T$$

$$\bar{v}(\varphi \wedge \psi) = \begin{cases} T & \text{if } \bar{v}(\varphi) = \bar{v}(\psi) = T \\ F & \text{otherwise} \end{cases}$$

$$\bar{v}(\neg\varphi) = \begin{cases} T & \text{if } \bar{v}(\varphi) = F \\ F & \text{otherwise} \end{cases}$$

$$\bar{v}(\varphi \rightarrow \psi) = \begin{cases} F & \text{if } \bar{v}(\varphi) = T \text{ and } \bar{v}(\psi) = F \\ T & \text{otherwise} \end{cases}$$

Example

$$v(p) = T \text{ and } v(q) = F \implies \bar{v}(p \wedge \neg q \rightarrow \neg p) = F$$

Definition

truth tables

φ	$\neg\varphi$	φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$
T	F	T	T	T	T	T
F	T	T	F	F	T	F
		F	T	F	T	T
		F	F	F	F	T

truth tables for propositional formulas are constructed bottom-up

Example 1

p	q	$\neg p$	$\neg q$	$p \rightarrow \neg q$	$q \vee \neg p$	$(p \rightarrow \neg q) \rightarrow (q \vee \neg p)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Example 2

p	q	$(p \rightarrow \neg q) \rightarrow (q \vee \neg p)$
T	T	T T
T	F	T T F F
F	T	T T T
F	F	T T T T

Definitions

► **semantic entailment**

$$\varphi_1, \varphi_2, \dots, \varphi_n \models \psi$$

if $\bar{v}(\psi) = T$ whenever $\bar{v}(\varphi_1) = \bar{v}(\varphi_2) = \dots = \bar{v}(\varphi_n) = T$ for every valuation v

► **tautology** is formula φ such that $\models \varphi$

Examples 1

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	
F	T	T	T
F	F	T	T

p	q	$p \rightarrow q, p \rightarrow \neg q$	$\neg p$
T	T	T	F
T	F	F	
F	T	T	T
F	F	T	T

Examples ②

p	q	$p \rightarrow q$	$\not\equiv$	$q \rightarrow p$	p	q	$p \rightarrow q, p \rightarrow \neg q$	$\not\equiv$	q
T	T	T		T	T	T	T		F
T	F	F			T	F	F		
F	T	T		F	F	T	T		T
F	F	T			F	F	T		F

p	q	$p \rightarrow q, p \wedge \neg q$	\equiv	\perp
T	T	T		F
T	F	F		
F	T	T		F
F	F	T		F

Question

$$\models (\neg p \wedge \neg q) \vee (s \wedge u) \vee (r \wedge w) \vee (\neg t \wedge \neg u) \vee (p \wedge r) \vee (q \wedge s) \\ \vee (p \wedge t) \vee (q \wedge u) \vee (\neg r \wedge \neg s) \vee (t \wedge v) \vee (\neg v \wedge \neg w) \quad ?$$

... truth table has $2^8 = 256$ rows ...

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

... truth table has $2^{459} > 2^{4 \times 100} = 16^{100} > 10^{100}$ rows ...

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Definitions

formula φ is

- ▶ **valid** if $\bar{v}(\varphi) = T$ for every valuation v
- ▶ **satisfiable** if $\bar{v}(\varphi) = T$ for some valuation v

Theorem

formula φ is valid $\iff \neg\varphi$ is unsatisfiable

Proof

$$\begin{aligned}\varphi \text{ is valid} &\iff \bar{v}(\varphi) = T \text{ for every valuation } v \\ &\iff \bar{v}(\neg\varphi) = F \text{ for every valuation } v \\ &\iff \bar{v}(\neg\varphi) = T \text{ for no valuation } v \\ &\iff \neg\varphi \text{ is not satisfiable} \iff \neg\varphi \text{ is unsatisfiable}\end{aligned}$$

Definition

formulas φ and ψ are **semantically equivalent** ($\varphi \equiv \psi$) if both $\varphi \models \psi$ and $\psi \models \varphi$

Examples

$$\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$$

$$\neg\neg\varphi \equiv \varphi$$

$$\varphi \vee (\psi \wedge \chi) \equiv (\varphi \vee \psi) \wedge (\varphi \vee \chi)$$

Theorem

validity and satisfiability are **decidable**

Proof

construct truth table of φ and inspect last column:

- ▶ φ is valid if and only if all entries are T
- ▶ φ is satisfiable if and only if T entry exists

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Question

Given that one and only one answer is correct, which of the following is true ?

- A** All of the below.
- B** None of the below.
- C** One of the above.
- D** All of the above.
- E** None of the above.
- F** None of the above.



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Definitions

- ▶ **literal** is atom p or negation $\neg p$ of atom
- ▶ **clause** is disjunction $l_1 \vee \dots \vee l_n$ of literals
- ▶ **conjunctive normal form (CNF)** is conjunction $C_1 \wedge \dots \wedge C_n$ of clauses
- ▶ literals l_1 and l_2 are **complementary** if $l_1 = \neg l_2$ or $\neg l_1 = l_2$

Theorem

validity of CNFs is **efficiently decidable**:

CNF φ is valid \iff every clause of φ contains **complementary literals**

Examples

1 CNF

$$(p \vee q \vee \neg r) \wedge (\neg p \vee \neg r \vee p) \wedge (\neg q)$$

complementary literals

not valid

witness: $v(p) = v(q) = F$ and $v(r) = T$

2 CNF

$$(p \vee q \vee \neg p) \wedge (\neg r \vee \neg p \vee r) \wedge (\neg q \vee q)$$

valid

Special Cases

- ▶ \perp represents empty clause (no literals)
- ▶ \top represents empty CNF (no clauses)

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Procedure

- ① eliminate implications
- ② push negations towards atoms and remove double negations
- ③ distribute disjunction over conjunction

$$\begin{array}{ll} \varphi \rightarrow \psi & \xrightarrow{\textcircled{1}} \neg\varphi \vee \psi \\ \neg(\varphi \wedge \psi) & \xrightarrow{\textcircled{2}} \neg\varphi \vee \neg\psi \\ \neg(\varphi \vee \psi) & \xrightarrow{\textcircled{2}} \neg\varphi \wedge \neg\psi \end{array} \qquad \begin{array}{ll} \neg\neg\varphi & \xrightarrow{\textcircled{2}} \varphi \\ \varphi \vee (\psi \wedge \chi) & \xrightarrow{\textcircled{3}} (\varphi \vee \psi) \wedge (\varphi \vee \chi) \\ (\varphi \wedge \psi) \vee \chi & \xrightarrow{\textcircled{3}} (\varphi \vee \chi) \wedge (\psi \vee \chi) \end{array}$$

Remark

CNF ψ for formula φ might be exponentially larger

Example (CNFs are not unique)

$$\varphi = \neg(p \vee (q \wedge r))$$

$$\xrightarrow{\textcircled{2}} \neg p \wedge \neg(q \wedge r) \xrightarrow{\textcircled{2}} \neg p \wedge (\neg q \vee \neg r)$$

$$\varphi = \neg(p \vee (q \wedge r))$$

$$\xrightarrow{\textcircled{3}} \neg((p \vee q) \wedge (p \vee r)) \xrightarrow{\textcircled{2}} \neg(p \vee q) \vee \neg(p \vee r)$$

$$\xrightarrow{\textcircled{2}} (\neg p \wedge \neg q) \vee \neg(p \vee r) \xrightarrow{\textcircled{2}} (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$$

$$\xrightarrow{\textcircled{3}} ((\neg p \wedge \neg q) \vee \neg p) \wedge ((\neg p \wedge \neg q) \vee \neg r)$$

$$\xrightarrow{\textcircled{3}} ((\neg p \vee \neg p) \wedge (\neg q \vee \neg p)) \wedge ((\neg p \wedge \neg q) \vee \neg r)$$

$$\xrightarrow{\textcircled{3}} ((\neg p \vee \neg p) \wedge (\neg q \vee \neg p)) \wedge ((\neg p \vee \neg r) \wedge (\neg q \vee \neg r))$$

CNFs are not unique, even if rules ①, ②, ③ are applied in order

Procedure (extended)

- ① simplify formulas with \perp and \top
- ① eliminate implications
- ② push negations towards atoms and remove double negations
- ③ distribute disjunction over conjunction

$$\neg \perp \xrightarrow{\textcircled{0}} \top$$

$$\perp \wedge \varphi \xrightarrow{\textcircled{0}} \perp$$

$$\perp \vee \varphi \xrightarrow{\textcircled{0}} \varphi$$

$$\perp \rightarrow \varphi \xrightarrow{\textcircled{0}} \top$$

$$\neg \top \xrightarrow{\textcircled{0}} \perp$$

$$\top \wedge \varphi \xrightarrow{\textcircled{0}} \varphi$$

$$\top \vee \varphi \xrightarrow{\textcircled{0}} \top$$

$$\top \rightarrow \varphi \xrightarrow{\textcircled{0}} \varphi$$

$$\varphi \wedge \perp \xrightarrow{\textcircled{0}} \perp$$

$$\varphi \vee \perp \xrightarrow{\textcircled{0}} \varphi$$

$$\varphi \rightarrow \perp \xrightarrow{\textcircled{0}} \neg \varphi$$

$$\varphi \wedge \top \xrightarrow{\textcircled{0}} \varphi$$

$$\varphi \vee \top \xrightarrow{\textcircled{0}} \top$$

$$\varphi \rightarrow \top \xrightarrow{\textcircled{0}} \top$$

Example

$$p \vee (q \wedge (\top \rightarrow (\neg p \vee \perp))) \rightarrow (\top \wedge \neg q)$$

$$\xrightarrow{\textcircled{0}} p \vee (q \wedge (\top \rightarrow \neg p) \rightarrow (\top \wedge \neg q)) \xrightarrow{\textcircled{0}} p \vee (q \wedge (\top \rightarrow \neg p) \rightarrow \neg q)$$

$$\xrightarrow{\textcircled{0}} p \vee (q \wedge \neg p \rightarrow \neg q) \xrightarrow{\textcircled{1}} p \vee (\neg(q \wedge \neg p) \vee \neg q)$$

$$\xrightarrow{\textcircled{2}} p \vee ((\neg q \vee \neg\neg p) \vee \neg q) \xrightarrow{\textcircled{2}} p \vee ((\neg q \vee p) \vee \neg q)$$

Example (CNF from truth table)

$$\varphi = \neg(p \vee (q \wedge r))$$

p	q	r	φ	p	q	r	φ
T	T	T	F	F	T	T	F
T	T	F	F	F	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	F	F	F	T

$$\begin{aligned}\varphi &\equiv \neg((p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r)) \\ &\equiv (\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee \neg q \vee \neg r)\end{aligned}$$

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

① eliminate implications

function IMPL_FREE(φ):

begin function

case φ is atom:	return φ
φ is $\neg\varphi_1$:	return \neg IMPL_FREE(φ_1)
φ is $\varphi_1 \wedge \varphi_2$:	return IMPL_FREE(φ_1) \wedge IMPL_FREE(φ_2)
φ is $\varphi_1 \vee \varphi_2$:	return IMPL_FREE(φ_1) \vee IMPL_FREE(φ_2)
φ is $\varphi_1 \rightarrow \varphi_2$:	return \neg IMPL_FREE(φ_1) \vee IMPL_FREE(φ_2)

end case

end function

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

② push negations towards atoms and remove double negations

function $\text{NNF}(\varphi)$:

begin function

case φ is literal:	return φ
φ is $\neg\neg\varphi_1$:	return $\text{NNF}(\varphi_1)$
φ is $\varphi_1 \wedge \varphi_2$:	return $\text{NNF}(\varphi_1) \wedge \text{NNF}(\varphi_2)$
φ is $\varphi_1 \vee \varphi_2$:	return $\text{NNF}(\varphi_1) \vee \text{NNF}(\varphi_2)$
φ is $\neg(\varphi_1 \wedge \varphi_2)$:	return $\text{NNF}(\neg\varphi_1) \vee \text{NNF}(\neg\varphi_2)$
φ is $\neg(\varphi_1 \vee \varphi_2)$:	return $\text{NNF}(\neg\varphi_1) \wedge \text{NNF}(\neg\varphi_2)$

end case

end function

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

③ distribute disjunction over conjunction

function CNF(φ):

begin function

case φ is literal: **return** φ

φ is $\varphi_1 \wedge \varphi_2$: **return** $\text{CNF}(\varphi_1) \wedge \text{CNF}(\varphi_2)$

φ is $\varphi_1 \vee \varphi_2$: **return** **DISTR**($\text{CNF}(\varphi_1), \text{CNF}(\varphi_2)$)

end case

end function

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

③ distribute disjunction over conjunction

function DISTR(η_1, η_2):

begin function

case η_1 is $\eta_{11} \wedge \eta_{12}$: **return** DISTR(η_{11}, η_2) \wedge DISTR(η_{12}, η_2)

η_2 is $\eta_{21} \wedge \eta_{22}$: **return** DISTR(η_1, η_{21}) \wedge DISTR(η_1, η_{22})

otherwise: **return** $\eta_1 \vee \eta_2$

end case

end function

Theorem

for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$

Deterministic Procedure

- ① eliminate implications
- ② push negations towards atoms and remove double negations
- ③ distribute disjunction over conjunction

Theorem

- ① CNF(NNF(IMPL_FREE(φ))) is CNF
- ② CNF(NNF(IMPL_FREE(φ))) $\equiv \varphi$
- ③ executing CNF(NNF(IMPL_FREE(φ))) terminates

Outline

1. Introduction
2. Propositional Logic
3. Satisfiability and Validity
4. Intermezzo
5. Conjunctive Normal Forms
- 6. Further Reading**

Huth and Ryan

- ▶ Section 1.1
- ▶ Section 1.3
- ▶ Sections 1.4.1 and 1.4.2
- ▶ Sections 1.5.1 and 1.5.2

Differences (slides – book)

- ▶ role of \perp and \top
- ▶ terminology concerning CNFs

Important Concepts

- ▶ atom
- ▶ bottom
- ▶ clause
- ▶ complementary literals
- ▶ conjunction
- ▶ conjunctive normal form
- ▶ disjunction
- ▶ disjunctive normal form
- ▶ implication
- ▶ literal
- ▶ negation
- ▶ top
- ▶ right-associativity
- ▶ satisfiability
- ▶ semantic entailment
- ▶ semantic equivalence
- ▶ tautology
- ▶ truth table
- ▶ truth values
- ▶ validity
- ▶ valuation

homework for **March 7** (this week)