SS 2024 lecture 2

Logic

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with session ID **0992 9580** for anonymous questions

parallel registration for VO and TU enabled

Outline

1. **Summary of Previous Lecture**
2. **Horn Formulas**
3. **Intermezzo**
4. **SAT**
5. **Tseitin’s Transformation**
6. **Further Reading**

Definitions

- semantic entailment

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi$$

  if $\bar{v}(\psi) = \mathrm{T}$ whenever $\bar{v}(\varphi_1) = \bar{v}(\varphi_2) = \cdots = \bar{v}(\varphi_n) = \mathrm{T}$ for every valuation v
- tautology is formula φ such that $\vDash \varphi$
- formula φ is
  - valid if $\bar{v}(\varphi) = \mathrm{T}$ for every valuation v
  - satisfiable if $\bar{v}(\varphi) = \mathrm{T}$ for some valuation v
- formulas φ and ψ are semantically equivalent $(\varphi \equiv \psi)$ if both $\varphi \vDash \psi$ and $\psi \vDash \varphi$

Theorem

formula φ is valid $\iff$ $\neg\varphi$ is unsatisfiable $\iff$ φ is tautology

Definitions

- literal is atom p or negation $\neg p$ of atom
- clause is disjunction $\ell_1 \vee \cdots \vee \ell_n$ of literals
- conjunctive normal form (CNF) is conjunction $C_1 \wedge \cdots \wedge C_n$ of clauses
- literals ℓ_1 and ℓ_2 are complementary if $\ell_1 = \neg\ell_2$ or $\neg\ell_1 = \ell_2$

Theorem

- for every formula φ there exists CNF ψ such that $\varphi \equiv \psi$
- validity of CNFs is efficiently decidable:

$$\text{CNF } \varphi \text{ is valid} \quad \Longleftrightarrow \quad \text{every clause of } \varphi \text{ contains complementary literals}$$

Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, soundness and completeness, syntax, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching-time temporal logic, CTL*, fairness, linear-time temporal logic, model checking algorithms, symbolic model checking

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Definitions

- Horn clause is propositional formula

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n \rightarrow Q$$

with $n \geqslant 1$ and where $P_1, \ldots, P_n, Q$ are atoms, $\bot$ or $\top$

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Backus–Naur Form (H)

$$\begin{aligned} P &::= p \mid \bot \mid \top & \qquad C &::= A \to P \\ A &::= P \mid P \wedge A & \qquad {\color{red}H} &::= C \mid C \wedge H \end{aligned}$$

Theorem

satisfiability of Horn formulas is efficiently decidable

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Procedure

① maintain list of atoms, $\bot$, $\top$ occurring in φ

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② mark $\top$ if it appears in list

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③ **while** Horn clause $P_1 \wedge \cdots \wedge P_n \to Q$ exists in φ such that all $P_1, \ldots, P_n$ are marked and Q is unmarked

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mark Q

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④ if $\bot$ is marked then

     return unsatisfiable

   else

     return satisfiable
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satisfying assignment: $v(P) = \begin{cases} \mathsf{T} & \text{if } P \text{ is marked} \\ \mathsf{F} & \text{if } P \text{ is unmarked} \end{cases}$

Examples

① Horn formula

$$(p \wedge q \wedge w \to \bot) \wedge (t \to \bot) \wedge (r \to p) \wedge (\top \to r) \wedge (\top \to q) \wedge (\top \to u) \wedge (u \to s)$$

list $p \quad q \quad r \quad s \quad t \quad u \quad w \quad \bot \quad \top$

Examples

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unsatisfiable

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Question

Consider the formula $\varphi = (p \wedge \neg q \to \bot) \wedge (q \wedge p \to \neg q)$.

Which of the following statements hold for φ ?

A φ is a CNF

B φ is a Horn formula

C $\varphi \equiv p \to \neg q$

D φ is satisfiable

E φ is valid

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instance: propositional formula φ

question: is φ satisfiable ?

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SAT is NP-complete

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Links

- SAT competition

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Links

- SAT competition
- Millennium Problems – P vs NP

SAT Applications

- bounded model checking
- combinatorial design theory
- haplotyping in bioinformatics
- hardware verification
- logic puzzles
- package management in software distributions
- planning and scheduling
- software verification
- sorting networks
- statistical physics
- term rewriting
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Popular SAT Solvers

MiniSat PicoSAT Z3

Example（数独　Sudoku）

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

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11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
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51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
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81	82	83	84	85	86	87	88	89
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Variables

- propositional atoms x_{ijd} for $i, j, d \in \{1, \ldots, 9\}$

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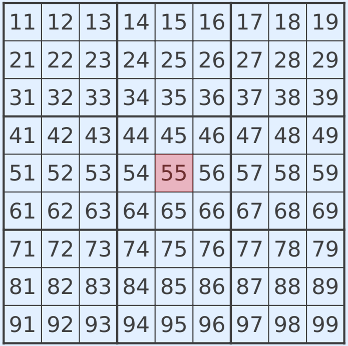
Variables

- propositional atoms x_{ijd} for $i, j, d \in \{1, \ldots, 9\}$
- $v(x_{ijd}) = \mathrm{T} \quad \Longleftrightarrow \quad$ cell ij contains digit d



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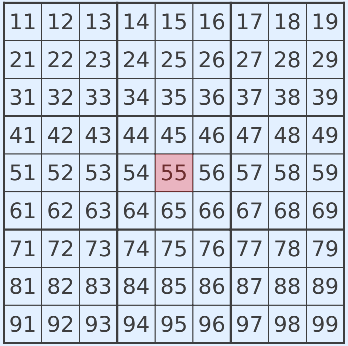
Constraints

- every cell contains at least one digit



11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
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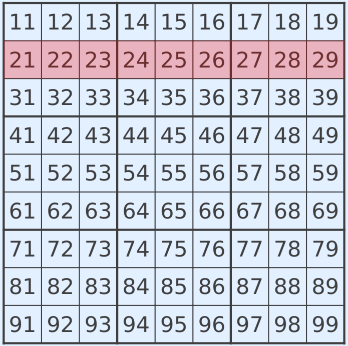

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21	22	23	24	25	26	27	28	29
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Constraints

- every cell contains at least one digit
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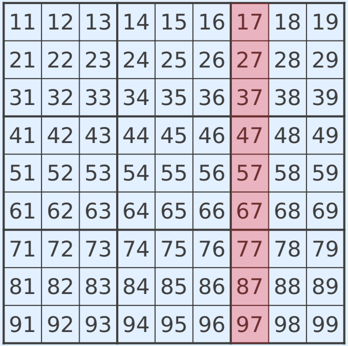

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
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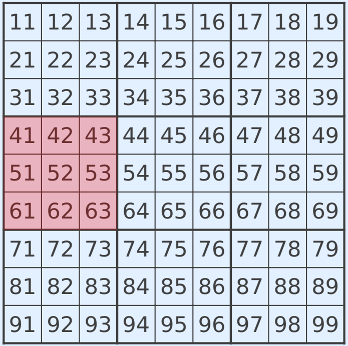

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21	22	23	24	25	26	27	28	29
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for non-empty set A of propositional atoms:

$$\text{at-least-one}(A) = \bigvee_{x \in A} x$$

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11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

SAT Encoding

$$\varphi\colon \quad \bigwedge\{\text{at-least-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\}$$

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

SAT Encoding

$$\begin{aligned}\varphi: \quad & \bigwedge \{\text{at-least-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\} \land \\ & \bigwedge \{\text{at-most-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\}\end{aligned}$$

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
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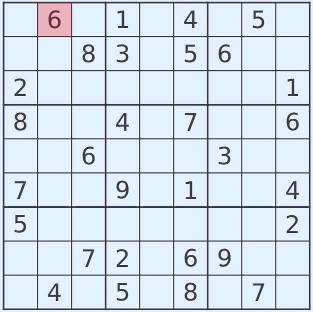
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

SAT Encoding

$$\varphi: \quad \bigwedge \{\text{at-least-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\} \land \bigwedge \{\text{at-most-one}(A) \mid A \in \mathcal{C}\} \land$$
$$\bigwedge \{\text{at-most-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\}$$

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99

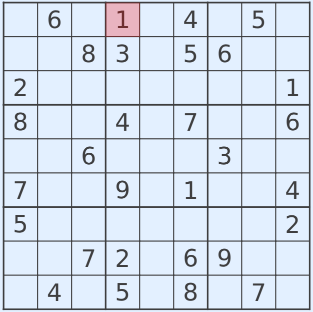


	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

SAT Encoding

$$\varphi: \quad \bigwedge \{\text{at-least-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\} \wedge \bigwedge \{\text{at-most-one}(A) \mid A \in \mathcal{C}\} \wedge$$
$$\bigwedge \{\text{at-most-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\} \wedge x_{126}$$

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99



	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

SAT Encoding

$$\varphi: \quad \bigwedge \{ \text{at-least-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D \} \wedge \bigwedge \{ \text{at-most-one}(A) \mid A \in \mathcal{C} \} \wedge$$
$$\bigwedge \{ \text{at-most-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D \} \wedge x_{126} \wedge x_{141}$$

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
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	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

SAT Encoding

$$\varphi: \quad \bigwedge \{\text{at-least-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\} \wedge \bigwedge \{\text{at-most-one}(A) \mid A \in \mathcal{C}\} \wedge$$

$$\bigwedge \{\text{at-most-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\} \wedge x_{126} \wedge x_{141} \wedge x_{164}$$

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
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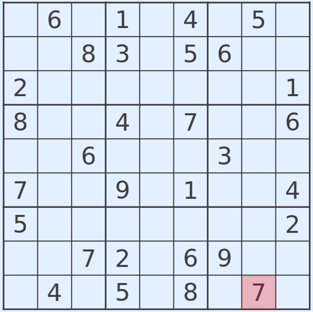

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

SAT Encoding

$$\varphi: \quad \bigwedge \{\text{at-least-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\} \wedge \bigwedge \{\text{at-most-one}(A) \mid A \in \mathcal{C}\} \wedge$$
$$\bigwedge \{\text{at-most-one}(\{x_{ijd} \mid d \in D\}) \mid i, j \in D\} \wedge x_{126} \wedge x_{141} \wedge x_{164} \wedge \cdots \wedge x_{987}$$

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
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91	92	93	94	95	96	97	98	99

	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

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- φ is satisfiable $\iff$ Sudoku puzzle has solution

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
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	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

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- φ is satisfiable $\iff$ Sudoku puzzle has solution
- satisfying assignment gives rise to Sudoku solution

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
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	6		1		4		5	
		8	3		5	6		
2								1
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

SAT Encoding

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- satisfying assignment gives rise to Sudoku solution

$$D = \{1,2,3,4,5,6,7,8,9\}$$

$$\mathcal{G} = \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$$

$$\mathcal{C} = \{\{x_{ijd} \mid j \in D\} \mid i,\, d \in D\} \cup \{\{x_{ijd} \mid i \in D\} \mid j,\, d \in D\} \cup \{\{x_{ijd} \mid (i,j) \in I \times J\} \mid I, J \in \mathcal{G},\, d \in D\}$$

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Example（2 × 2 数独　Sudoku）

		1	
3			
	4		

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

$$D = \{1, 2, 3, 4\}$$

$$\mathcal{G} = \{\{1, 2\}, \{3, 4\}\}$$

$$\mathcal{C} = \{\{x_{ijd} \mid j \in D\} \mid i, d \in D\} \cup \{\{x_{ijd} \mid i \in D\} \mid j, d \in D\} \cup \{\{x_{ijd} \mid (i, j) \in I \times J\} \mid I, J \in \mathcal{G}, d \in D\}$$

Example（2 × 2 数独　Sudoku）

		1	
3			
	4		

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

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Example (2 × 2 数独 Sudoku)

		1	
3			
	4		

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

$$\mathcal{C} = \{\{x_{111}, x_{121}, x_{131}, x_{141}\}, \{x_{112}, x_{122}, x_{132}, x_{142}\}, \ldots, \{x_{414}, x_{424}, x_{434}, x_{444}\}\}$$

$$D = \{1,2,3,4\}$$

$$\mathcal{G} = \{\{1,2\},\{3,4\}\}$$

$$\mathcal{C} = \{\{x_{ijd} \mid j \in D\} \mid i,\, d \in D\} \cup \{\{x_{ijd} \mid i \in D\} \mid j,\, d \in D\} \cup \{\{x_{ijd} \mid (i,j) \in I \times J\} \mid I, J \in \mathcal{G},\, d \in D\}$$

Example（2 × 2 数独　**Sudoku**）

		1	
3			
	4		

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21	22	23	24
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Pythagorean Triples Problem

can one color all natural numbers with two colors such that whenever $x^2 + y^2 = z^2$ not all of x, y, z have same color ?

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- encoding contains clauses $(x_a \vee x_b \vee x_c)$ and $(\neg x_a \vee \neg x_b \vee \neg x_c)$ for all $a^2 + b^2 = c^2$

Solution

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- further details

Outline

1. **Summary of Previous Lecture**
2. **Horn Formulas**
3. **Intermezzo**
4. **SAT**
5. **Tseitin’s Transformation**
6. **Further Reading**

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most SAT solvers require CNF as input

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4 atoms

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4 atoms and 3 clauses

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formulas φ and ψ are equisatisfiable ($\varphi \approx \psi$) if

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$$(p \vee q) \wedge \neg p \approx \top$$

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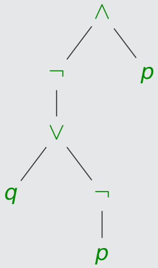
Examples

$$(p \vee q) \wedge \neg p \approx \top \qquad\qquad (p \vee q) \wedge \neg p \not\approx q \wedge \neg q$$

Example (Tseitin's Transformation)

- $\varphi = \neg(q \vee \neg p) \wedge p$



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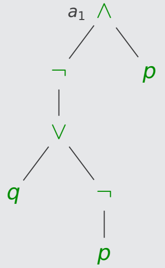
- $\varphi = \neg(q \vee \neg p) \wedge p$
- introduce new variable for each propositional connective:

$$a_1 \quad \neg(q \vee \neg p) \wedge p$$



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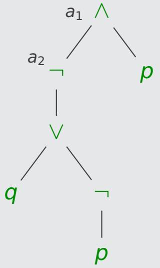

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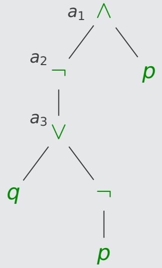

a_1	$\neg(q \vee \neg p) \wedge p$	a_3	$q \vee \neg p$
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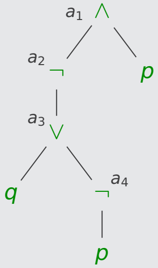

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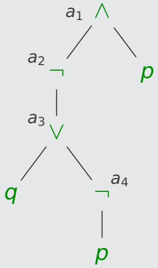

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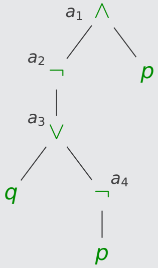

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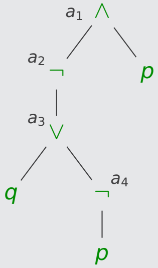

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new propositional connective

- equivalence $\leftrightarrow$ $p \leftrightarrow q$ "p is equivalent to q"
- $\bar{v}(\varphi \leftrightarrow \psi) = \begin{cases} \mathsf{T} & \text{if } \bar{v}(\varphi) = \bar{v}(\psi) \\ \mathsf{F} & \text{otherwise} \end{cases}$

Notational Convention

binding precedence $\neg \quad > \quad \wedge, \vee \quad > \quad \rightarrow, \leftrightarrow$

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Proof

φ	ψ	$\varphi \leftrightarrow \psi$	$(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
T	T	T	T
T	F	F	F
F	T	F	F
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Lemma

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①

φ	ψ	$\varphi \leftrightarrow \neg\psi$	$(\varphi \vee \psi) \wedge (\neg\varphi \vee \neg\psi)$
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\varphi &\approx a_1 \wedge (a_1 \leftrightarrow a_2 \wedge p) \wedge (a_2 \leftrightarrow \neg a_3) \wedge (a_3 \leftrightarrow q \vee a_4) \wedge (a_4 \leftrightarrow \neg p) \\
&\equiv a_1 \wedge (\neg a_1 \vee a_2) \wedge (\neg a_1 \vee p) \wedge (a_1 \vee \neg a_2 \vee \neg p) \wedge (a_2 \vee a_3) \wedge (\neg a_2 \vee \neg a_3) \\
&\qquad \wedge (a_3 \vee \neg q) \wedge (a_3 \vee \neg a_4) \wedge (\neg a_3 \vee q \vee a_4) \wedge (a_4 \vee p) \wedge (\neg a_4 \vee \neg p)
\end{aligned}$$

Definition (Tseitin's Transformation)

for propositional formula φ

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Lemma

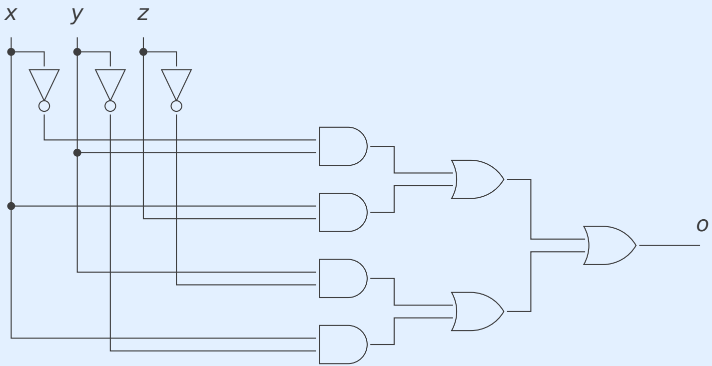
① any satisfying valuation for φ can be (uniquely) extended to satisfying valuation for $\mathsf{TT}(\varphi)$

Lemma

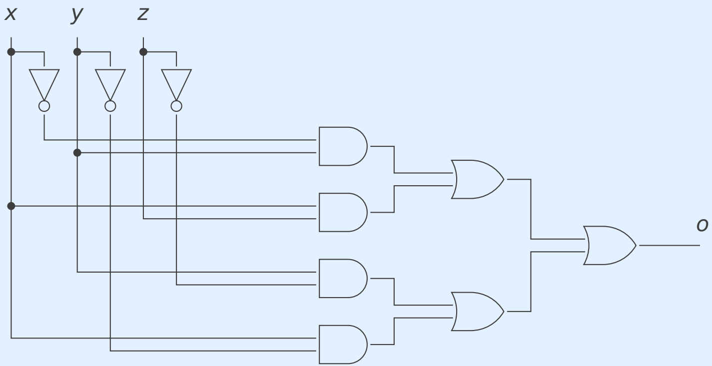
① any satisfying valuation for φ can be (uniquely) extended to satisfying valuation for $\mathsf{TT}(\varphi)$

② restriction of any satisfying valuation for $\mathsf{TT}(\varphi)$ to atoms in φ is satisfying valuation for φ

Logic Circuit



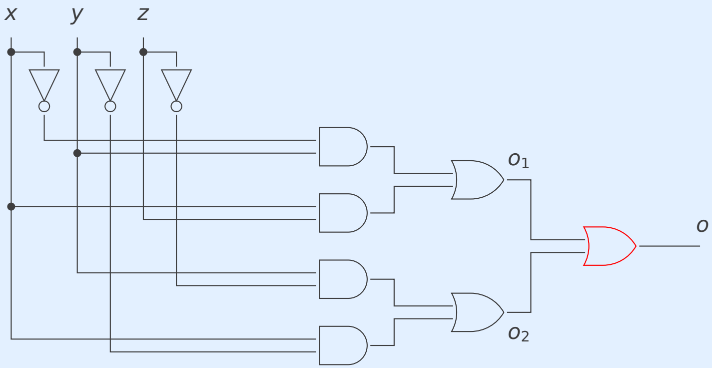
Logic Circuit



Equisatisfiable CNF

o

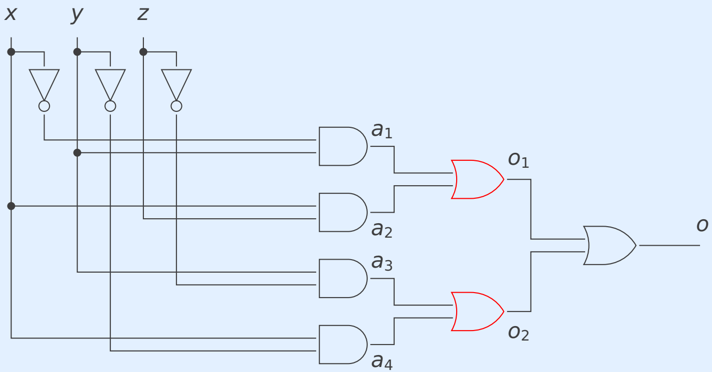
Logic Circuit



Equisatisfiable CNF

$$o \land (o \leftrightarrow o_1 \lor o_2)$$

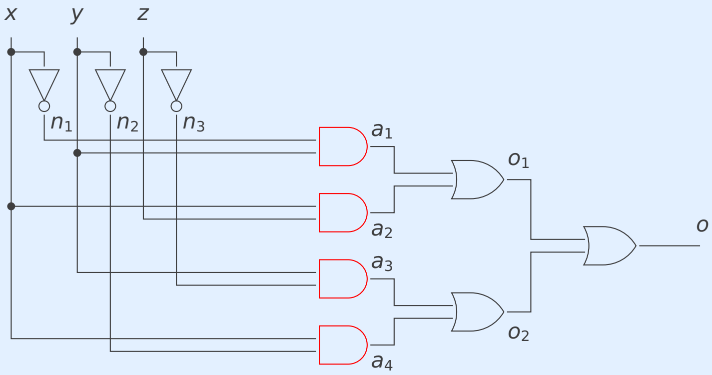
Logic Circuit



Equisatisfiable CNF

$$o \land (o \leftrightarrow o_1 \lor o_2) \land (o_1 \leftrightarrow a_1 \lor a_2) \land (o_2 \leftrightarrow a_3 \lor a_4)$$

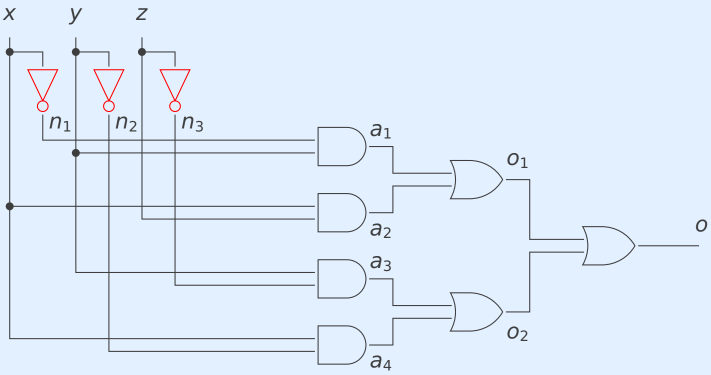
Logic Circuit



Equisatisfiable CNF

$$o \wedge (o \leftrightarrow o_1 \vee o_2) \wedge (o_1 \leftrightarrow a_1 \vee a_2) \wedge (o_2 \leftrightarrow a_3 \vee a_4) \wedge (a_1 \leftrightarrow n_1 \wedge y) \wedge (a_2 \leftrightarrow x \wedge z)$$
$$\wedge (a_3 \leftrightarrow y \wedge n_3) \wedge (a_4 \leftrightarrow x \wedge n_2)$$

Logic Circuit



Equisatisfiable CNF

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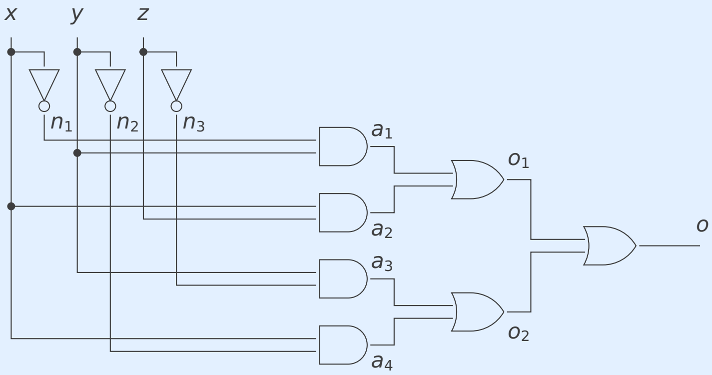
$$\wedge (a_3 \leftrightarrow y \wedge n_3) \wedge (a_4 \leftrightarrow x \wedge n_2) \wedge (n_1 \leftrightarrow \neg x) \wedge (n_2 \leftrightarrow \neg y) \wedge (n_3 \leftrightarrow \neg z)$$

Logic Circuit



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Outline

1. **Summary of Previous Lecture**
2. **Horn Formulas**
3. **Intermezzo**
4. **SAT**
5. **Tseitin's Transformation**
6. **Further Reading**

Huth and Ryan

- Section 1.5

Huth and Ryan

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SAT and P – NP

- SAT live! [accessed January 22, 2024]

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  doi: 10.1145/3107239

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  doi: 10.1145/3107239
- Fifty Years of P vs. NP and the Possibility of the Impossible
  Lance Fortnow
  Communications of the ACM 65(1), pp. 76 – 85, 2022
  doi: 10.1145/3460351

Important Concepts

- DIMACS format
- equisatisfiability
- equivalence
- Horn clause
- Horn formula
- SAT
- Tseitin's transformation

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homework for March 14