universität innsbruck

SS 2024 lecture 3

Logic

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Particify with session ID **0992 9580** for anonymous questions

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parallel registration for VO and TU enabled

Outline

Definition

- Horn clause is propositional formula

$$P_1 \wedge P_2 \wedge \cdots \wedge P_n \rightarrow Q$$

  with $n \geqslant 1$ and where $P_1, \ldots, P_n, Q$ are atoms, $\bot$ or $\top$
- Horn formula is conjunction of Horn clauses

Theorem

satisfiability of Horn formulas is efficiently decidable

Remark

deciding satisfiability for arbitrary formulas is important and difficult problem (SAT)

Definition

formulas φ and ψ are equisatisfiable ($\varphi \approx \psi$) if

$$\varphi \text{ is satisfiable} \iff \psi \text{ is satisfiable}$$

Remark

Tseitin's transformation transforms arbitrary formula into equisatisfiable CNF in linear time

Lemma

1. any satisfying valuation for φ can be (uniquely) extended to satisfying valuation for $\mathsf{TT}(\varphi)$
2. restriction of any satisfying valuation for $\mathsf{TT}(\varphi)$ to atoms in φ is satisfying valuation for φ

Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, soundness and completeness, syntax, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching-time temporal logic, CTL*, fairness, linear-time temporal logic, model checking algorithms, symbolic model checking

Outline

Natural Deduction

calculus for reasoning about propositions

Definitions

- sequent

$$\underbrace{\varphi_1, \varphi_2, \ldots, \varphi_n}_{\text{premises}} \vdash \underbrace{\psi}_{\text{conclusion}}$$

  with propositional formulas $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi$
- sequent $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is valid if ψ can be proved from premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ using proof rules of natural deduction

natural deduction consists of 17 proof rules

Definition

- and introduction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \wedge\mathrm{i}$$

Example

$p, q, r \vdash (r \wedge q) \wedge p$ is valid:

1	p	premise
2	q	premise
3	r	premise
4	$r \wedge q$	$\wedge$i 3, 2
5	$(r \wedge q) \wedge p$	$\wedge$i 4, 1

Definition

- and elimination

$$\frac{\varphi \wedge \psi}{\varphi} \wedge\mathrm{e}_1 \qquad \frac{\varphi \wedge \psi}{\psi} \wedge\mathrm{e}_2$$

Example

$p \wedge q, r \vdash r \wedge q$ is valid:

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge\mathrm{e}_2$ 1
4	$r \wedge q$	$\wedge$i 2, 3

Definitions

- double negation elimination $\frac{\neg\neg\varphi}{\varphi} \neg\neg\mathrm{e}$

- double negation introduction $\frac{\varphi}{\neg\neg\varphi} \neg\neg\mathrm{i}$

Example

$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$ is valid:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg$i 1
4	$q \wedge r$	$\neg\neg$e 2
5	r	$\wedge\mathrm{e}_2$ 4
6	$\neg\neg p \wedge r$	$\wedge$i 3, 5

Definition

- implication elimination (modus ponens)

$$\frac{\varphi \to \psi \quad \varphi}{\psi} \to\mathrm{e}$$

Example

$p, p \to q, p \to (q \to r) \vdash r$ is valid:

1	p	premise
2	$p \to q$	premise
3	$p \to (q \to r)$	premise
4	q	$\to$e 2, 1
5	$q \to r$	$\to$e 3, 1
6	r	$\to$e 5, 4

Definition

- modus tollens

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}\ \text{MT}$$

Example

$\neg p \to q,\ \neg q \vdash p$ is valid:

1	$\neg p \to q$	premise
2	$\neg q$	premise
3	$\neg\neg p$	MT 1, 2
4	p	$\neg\neg$e 3

Definition

- implication introduction

box

$$\frac{\boxed{\begin{array}{c}\varphi \\ \vdots \\ \psi\end{array}}}{\varphi \to \psi}\ \to\text{i}$$

assumption

Example

$\neg q \to \neg p \vdash p \to q$ is valid:

1	$\neg q \to \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg$i 2
4	$\neg\neg q$	MT 1, 3
5	q	$\neg\neg$e 4
6	$p \to q$	$\to$i 2–5

(lines 2–5 form a box)

Definition

- or introduction

$$\frac{\varphi}{\varphi \lor \psi}\ \lor\text{i}_1 \qquad\qquad \frac{\psi}{\varphi \lor \psi}\ \lor\text{i}_2$$

Example

$p \land q \vdash \neg q \lor p$ is valid:

1	$p \land q$	premise
2	p	$\land\text{e}_1$ 1
3	$\neg q \lor p$	$\lor\text{i}_2$ 2

Definition

- or elimination

$$\frac{\varphi \lor \psi \qquad \boxed{\begin{array}{c}\varphi \\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi}\ \lor\text{e}$$

Example

$p \lor q \vdash q \lor p$ is valid:

1	$p \lor q$	premise
2	p	assumption
3	$q \lor p$	$\lor\text{i}_2$ 2
4	q	assumption
5	$q \lor p$	$\lor\text{i}_1$ 4
6	$q \lor p$	$\lor$e 1, 2–3, 4–5

(lines 2–3 and lines 4–5 each form a box)

Definition

theorem is formula φ such that sequent $\vdash \varphi$ is valid

Example

$p \vee q \to q \vee p$ is theorem:

1	$p \vee q$	assumption
2	p	assumption
3	$q \vee p$	$\vee i_2$ 2
4	q	assumption
5	$q \vee p$	$\vee i_1$ 4
6	$q \vee p$	$\vee e$ 1, 2–3, 4–5
7	$p \vee q \to q \vee p$	$\to i$ 1–6

Definitions

- bottom elimination
  $$\frac{\bot}{\varphi}\ \bot e$$
- negation elimination
  $$\frac{\varphi \quad \neg\varphi}{\bot}\ \neg e$$
- negation introduction
  $$\frac{\boxed{\begin{matrix}\varphi \\ \vdots \\ \bot\end{matrix}}}{\neg\varphi}\ \neg i$$
- top introduction
  $$\frac{}{\top}\ \top i$$

Examples

$p \to q,\ p \to \neg q \vdash \neg p$ is valid:

1	$p \to q$	premise
2	$p \to \neg q$	premise
3	p	assumption
4	q	$\to e$ 1, 3
5	$\neg q$	$\to e$ 2, 3
6	$\bot$	$\neg e$ 4, 5
7	$\neg p$	$\neg i$ 3–6

$p,\ p \to q,\ p \to \neg q \vdash r$ is valid:

1	p	premise
2	$p \to q$	premise
3	$p \to \neg q$	premise
4	q	$\to e$ 2, 1
5	$\neg q$	$\to e$ 3, 1
6	$\bot$	$\neg e$ 4, 5
7	r	$\bot e$ 6

Example

$\top$ is theorem: 1 $\top$ $\top i$

Definitions

- proof by contradiction
  $$\frac{\boxed{\begin{matrix}\neg\varphi \\ \vdots \\ \bot\end{matrix}}}{\varphi}\ \text{PBC}$$
- law of excluded middle
  $$\frac{}{\varphi \vee \neg\varphi}\ \text{LEM}$$

Example

$p \rightarrow q \vee r,\ q \rightarrow \neg p,\ \neg r \rightarrow p \vdash q \rightarrow r$ is valid:

1	$p \rightarrow q \vee r$	premise
2	$q \rightarrow \neg p$	premise
3	$\neg r \rightarrow p$	premise
4	q	assumption
5	$\neg p$	$\rightarrow$e 2, 4
6	$\neg r$	assumption
7	p	$\rightarrow$e 3, 6
8	$\bot$	$\neg$e 7, 5
9	r	PBC 6–8
10	$q \rightarrow r$	$\rightarrow$i 4–9

Example

$p \rightarrow q \vdash \neg p \vee q$ is valid:

1	$p \rightarrow q$	premise
2	$p \vee \neg p$	LEM
3	p	assumption
4	q	$\rightarrow$e 1, 3
5	$\neg p \vee q$	$\vee$i$_2$ 4
6	$\neg p$	assumption
7	$\neg p \vee q$	$\vee$i$_1$ 6
8	$\neg p \vee q$	$\vee$e 2, 3–5, 6–7

Summary of Natural Deduction ❶

	introduction	elimination
$\wedge$	$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\varphi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\varphi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \psi\end{array}}}{\varphi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\varphi \rightarrow \psi \quad \varphi}{\psi}\ \rightarrow\mathrm{e}$

Summary of Natural Deduction ❷

	introduction	elimination
$\neg$	$\dfrac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \bot\end{array}}}{\neg\varphi}\ \neg\mathrm{i}$	$\dfrac{\varphi \quad \neg\varphi}{\bot}\ \neg\mathrm{e}$
$\bot$		$\dfrac{\bot}{\varphi}\ \bot\mathrm{e}$
$\top$	$\dfrac{}{\top}\ \top\mathrm{i}$	
$\neg\neg$		$\dfrac{\neg\neg\varphi}{\varphi}\ \neg\neg\mathrm{e}$

Summary of Natural Deduction ❸

derived proof rules

$$\frac{\varphi \to \psi \quad \neg\psi}{\neg\varphi}\ \text{MT} \qquad \frac{\varphi}{\neg\neg\varphi}\ \neg\neg\text{i} \qquad \frac{\boxed{\begin{array}{c}\neg\varphi\\ \vdots\\ \bot\end{array}}}{\varphi}\ \text{PBC} \qquad \frac{}{\varphi \vee \neg\varphi}\ \text{LEM}$$

Theorem

proof rules MT, $\neg\neg$i, PBC and LEM are derivable from other (basic) proof rules

Theorem

proof rules MT, $\neg\neg$i, PBC and LEM are derivable from other (basic) proof rules

Proof

1	$\varphi \to \psi$	premise
2	$\neg\psi$	premise
3	φ	assumption
4	ψ	$\to$e 1, 3
5	$\bot$	$\neg$e 4, 2
6	$\neg\varphi$	$\neg$i 3–5

Theorem

proof rules MT, $\neg\neg$i, PBC and LEM are derivable from other (basic) proof rules

Proof

1	φ	premise
2	$\neg\varphi$	assumption
3	$\bot$	$\neg$e 1, 2
4	$\neg\neg\varphi$	$\neg$i 2–3

Theorem

proof rules MT, $\neg\neg$i, PBC and LEM are derivable from other (basic) proof rules

Proof

1	$\neg\varphi$		hypothesis
	$\vdots$		
n	$\bot$		
$n+1$	$\neg\neg\varphi$	$\neg$i 1–n	
$n+2$	φ	$\neg\neg$e $n+1$	conclusion

Theorem

proof rules MT, $\neg\neg$i, PBC and LEM are derivable from other (basic) proof rules

Proof

1	$\neg(\varphi \vee \neg\varphi)$	assumption
2	φ	assumption
3	$\varphi \vee \neg\varphi$	$\vee i_1$ 2
4	$\bot$	$\neg$e 3, 1
5	$\neg\varphi$	$\neg$i 2–4
6	$\varphi \vee \neg\varphi$	$\vee i_2$ 5
7	$\bot$	$\neg$e 6, 1
8	$\neg\neg(\varphi \vee \neg\varphi)$	$\neg$i 1–7
9	$\varphi \vee \neg\varphi$	$\neg\neg$e 8

Theorem

proof rules LEM, PBC and $\neg\neg$e are inter-derivable (with respect to other basic proof rules)

Remark

- LEM, PBC and $\neg\neg$e are controversial because they are not constructive
- classical logicians use all proof rules
- intuitionistic logicians do not use LEM, PBC and $\neg\neg$e

Example

- formula $((p \to q) \to p) \to p$ is valid
- sequent $\vdash ((p \to q) \to p) \to p$ is valid but proof requires LEM, PBC or $\neg\neg$e

Outline

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Question

Which of the following statements are true ?

- **A** Intuitionistic logicians can prove more statements than classical logicians.
- **B** Every valid sequent has a proof without $\neg\neg$i.
- **C** Valid sequents can have arbitrarily long natural deduction proofs.
- **D** The sequent $p \to q \vdash \neg q \to \neg p$ is valid.
- **E** Every connective has an introduction rule as well as an elimination rule.

Outline

Theorem

natural deduction is sound:

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi \text{ is valid} \implies \varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi$$

only true statements can be proved

Corollary

theorems are valid

Idea

use induction on length of natural deduction proof and case analysis of last proof step

Problem

initial part of proof need not correspond to sequent with same premises

1	$p \to q$	premise	$p \to q, p \to \neg q \vdash p \to q$
2	$p \to \neg q$	premise	$p \to q, p \to \neg q \vdash p \to \neg q$
3	p	assumption	?
4	q	$\to$e 1, 3	?
5	$\neg q$	$\to$e 2, 3	?
6	$\bot$	$\neg$e 4, 5	?
7	$\neg p$	$\neg$i 3–6	$p \to q, p \to \neg q \vdash \neg p$

Solution

add assumptions to sequents

1	$p \to q$	premise	$p \to q, p \to \neg q \vdash p \to q$
2	$p \to \neg q$	premise	$p \to q, p \to \neg q \vdash p \to \neg q$
3	p	assumption	$p \to q, p \to \neg q\,; p \vdash p$
4	q	$\to$e 1, 3	$p \to q, p \to \neg q\,; p \vdash q$
5	$\neg q$	$\to$e 2, 3	$p \to q, p \to \neg q\,; p \vdash \neg q$
6	$\bot$	$\neg$e 4, 5	$p \to q, p \to \neg q\,; p \vdash \bot$
7	$\neg p$	$\neg$i 3–6	$p \to q, p \to \neg q \vdash \neg p$

Definition

extended sequent

$$\underbrace{\varphi_1, \varphi_2, \ldots, \varphi_n}_{\substack{\text{premises} \\ \Phi_1}}\,;\,\underbrace{\psi_1, \psi_2, \ldots, \psi_m}_{\substack{\text{assumptions} \\ \Phi_2}} \vdash \underbrace{\chi}_{\text{conclusion}}$$

Example

1	$p \wedge q \to r$	premise	$p \wedge q \to r \vdash p \wedge q \to r$
2	p	assumption	$p \wedge q \to r\,;\, p \vdash p$
3	q	assumption	$p \wedge q \to r\,;\, p,\, q \vdash q$
4	$p \wedge q$	$\wedge$i 2, 3	$p \wedge q \to r\,;\, p,\, q \vdash p \wedge q$
5	r	$\to$e 1, 4	$p \wedge q \to r\,;\, p,\, q \vdash r$
6	$q \to r$	$\to$i 3–5	$p \wedge q \to r\,;\, p \vdash q \to r$
7	$p \to (q \to r)$	$\to$i 2–6	$p \wedge q \to r \vdash p \to (q \to r)$

Soundness Proof

by induction on length of proof of

$$\Phi_1;\, \Phi_2 \vdash \psi$$

we prove

$$\Phi_1,\, \Phi_2 \vDash \psi$$

Claim

$\Phi_1;\, \Phi_2 \vdash \psi$ is valid $\implies$ $\Phi_1,\, \Phi_2 \vDash \psi$

Base Cases

- premise $\psi \in \Phi_1 \implies \Phi_1,\, \Phi_2 \vDash \psi$
- assumption $\psi \in \Phi_2 \implies \Phi_1,\, \Phi_2 \vDash \psi$
- $\top$i $\psi = \top \implies \Phi_1,\, \Phi_2 \vDash \psi$

Claim

$\Phi_1;\, \Phi_2 \vdash \psi$ is valid $\implies$ $\Phi_1,\, \Phi_2 \vDash \psi$

Induction Step (case analysis of last proof step) $\wedge$i

$\psi = \psi_1 \wedge \psi_2$

shorter proofs $\Phi_1;\, \Phi_2^1 \vdash \psi_1$ and $\Phi_1;\, \Phi_2^2 \vdash \psi_2$ with $\Phi_2^1,\, \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1,\, \Phi_2^1 \vDash \psi_1$ and $\Phi_1,\, \Phi_2^2 \vDash \psi_2$

$$\begin{aligned} \bar{v}(\varphi) = \mathsf{T} \text{ for all } \varphi \in \Phi_1,\, \Phi_2 \quad &\implies \quad \bar{v}(\varphi) = \mathsf{T} \text{ for all } \varphi \in \Phi_1,\, \Phi_2^1,\, \Phi_2^2 \\ &\implies \quad \bar{v}(\psi_1) = \bar{v}(\psi_2) = \mathsf{T} \\ &\implies \quad \bar{v}(\psi_1 \wedge \psi_2) = \mathsf{T} \end{aligned}$$

hence $\Phi_1,\, \Phi_2 \vDash \psi$

Claim

$\Phi_1;\, \Phi_2 \vdash \psi$ is valid $\implies$ $\Phi_1,\, \Phi_2 \vDash \psi$

Induction Step (case analysis of last proof step) $\to$i

$\psi = \psi_1 \to \psi_2$

shorter proof $\Phi_1;\, \Phi_2^1,\, \psi_1 \vdash \psi_2$ with $\Phi_2^1 \subseteq \Phi_2$

induction hypothesis: $\Phi_1,\, \Phi_2^1,\, \psi_1 \vDash \psi_2$

$\bar{v}(\varphi) = \mathsf{T}$ for all $\varphi \in \Phi_1,\, \Phi_2$

two cases

$\bar{v}(\psi_1) = \mathsf{F} \implies \bar{v}(\psi_1 \to \psi_2) = \mathsf{T}$

$\bar{v}(\psi_1) = \mathsf{T} \implies \bar{v}(\varphi) = \mathsf{T}$ for all $\varphi \in \Phi_1,\, \Phi_2^1,\, \psi_1 \implies \bar{v}(\psi_2) = \mathsf{T} \implies \bar{v}(\psi_1 \to \psi_2) = \mathsf{T}$

hence $\Phi_1,\, \Phi_2 \vDash \psi$

Claim

$\Phi_1; \Phi_2 \vdash \psi$ is valid $\implies$ $\Phi_1, \Phi_2 \vDash \psi$

Induction Step (case analysis of last proof step) $\neg$e

$\psi = \bot$

shorter proofs $\Phi_1; \Phi_2^1 \vdash \psi'$ and $\Phi_1; \Phi_2^2 \vdash \neg\psi'$ with $\Phi_2^1, \Phi_2^2 \subseteq \Phi_2$

induction hypothesis: $\Phi_1, \Phi_2^1 \vDash \psi'$ and $\Phi_1, \Phi_2^2 \vDash \neg\psi'$

$$\begin{aligned} \bar{v}(\varphi) = \mathsf{T} \text{ for all } \varphi \in \Phi_1, \Phi_2 \quad &\implies \quad \bar{v}(\varphi) = \mathsf{T} \text{ for all } \varphi \in \Phi_1, \Phi_2^1, \Phi_2^2 \\ &\implies \quad \bar{v}(\psi') = \mathsf{T} \text{ and } \bar{v}(\neg\psi') = \mathsf{T} \quad ϟ \end{aligned}$$

hence $\Phi_1, \Phi_2 \vDash \bot$

Claim

$\Phi_1; \Phi_2 \vdash \psi$ is valid $\implies$ $\Phi_1, \Phi_2 \vDash \psi$

Induction Step (case analysis of last proof step) $\vee$e

shorter proofs $\Phi_1; \Phi_2^1 \vdash \psi_1 \vee \psi_2$ and $\Phi_1; \Phi_2^2, \psi_1 \vdash \psi$ and $\Phi_1; \Phi_2^3, \psi_2 \vdash \psi$ with $\Phi_2^1, \Phi_2^2, \Phi_2^3 \subseteq \Phi_2$

induction hypothesis: $\Phi_1, \Phi_2^1 \vDash \psi_1 \vee \psi_2$ and $\Phi_1; \Phi_2^2, \psi_1 \vDash \psi$ and $\Phi_1; \Phi_2^3, \psi_2 \vDash \psi$

$\bar{v}(\varphi) = \mathsf{T}$ for all $\varphi \in \Phi_1, \Phi_2$ $\implies$ $v(\psi_1 \vee \psi_2) = \mathsf{T}$

two cases

$$\begin{aligned} \bar{v}(\psi_1) = \mathsf{T} \quad &\implies \quad \bar{v}(\psi) = \mathsf{T} \\ \bar{v}(\psi_2) = \mathsf{T} \quad &\implies \quad \bar{v}(\psi) = \mathsf{T} \end{aligned}$$

hence $\Phi_1, \Phi_2 \vDash \psi$

Claim

$\Phi_1; \Phi_2 \vdash \psi$ is valid $\implies$ $\Phi_1, \Phi_2 \vDash \psi$

Induction Step (case analysis of last proof step)

remaining (basic) proof rules

$$\wedge e_1 \quad \wedge e_2 \quad \vee i_1 \quad \vee i_2 \quad \rightarrow e \quad \neg i \quad \bot e \quad \neg\neg e$$

are similar

Corollary

natural deduction is sound

Outline

Huth and Ryan

- Section 1.2
- Sections 1.4.2 and 1.4.3

Intuitionistic Logic

- Wikipedia [accessed January 23, 2024]

Differences (slides – book)

- top introduction proof rule
- order of premises in $\to$e

Important Concepts

- and elimination
- and introduction
- assumption
- bottom elimination
- derived proof rule
- double negation introduction
- double negation elimination
- extended sequent
- implication elimination
- implication introduction
- intuitionistic logic
- law of excluded middle
- modus ponens
- modus tollens
- natural deduction
- negation elimination
- negation introduction
- or elimination
- or introduction
- premise
- proof by contradiction
- sequent
- soundness
- theorem
- top introduction
- validity of sequents

homework for March 21