Logic

Diana Gründlinger Aart Middeldorp Fabian Mitterwallner

Alexander Montag Johannes Niederhauser Daniel Rainer

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with session ID **0992 9580** for anonymous questions

Outline

Definitions

- sequent

$$\underbrace{\varphi_1, \varphi_2, \ldots, \varphi_n}_{\text{premises}} \vdash \underbrace{\psi}_{\text{conclusion}}$$

  with propositional formulas $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi$

- sequent $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is valid if ψ can be proved from premises $\varphi_1, \varphi_2, \ldots, \varphi_n$ using proof rules of natural deduction
- theorem is formula φ such that sequent $\vdash \varphi$ is valid

Theorem

- natural deduction is sound: $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is valid $\implies$ $\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi$
- proof rules MT, $\neg\neg$i, PBC and LEM are derivable from basic proof rules
- proof rules LEM, PBC and $\neg\neg$e are inter-derivable (with respect to other basic proof rules)

Proof Rules of Natural Deduction ❶

	introduction	elimination
$\wedge$	$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\varphi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\varphi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \psi\end{array}}}{\varphi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\varphi \rightarrow \psi \quad \varphi}{\psi}\ \rightarrow\mathrm{e}$

Proof Rules of Natural Deduction ❷

	introduction	elimination
$\bot$		$\dfrac{\bot}{\varphi}\ \bot e$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \bot\end{array}}}{\neg\varphi}\ \neg i$	$\dfrac{\varphi \quad \neg\varphi}{\bot}\ \neg e$
$\top$	$\dfrac{}{\top}\ \top i$	
$\neg\neg$		$\dfrac{\neg\neg\varphi}{\varphi}\ \neg\neg e$

derived proof rules

$$\frac{\varphi \to \psi \quad \neg\psi}{\neg\varphi}\ \text{MT} \qquad \frac{\boxed{\begin{array}{c}\neg\varphi\\ \vdots\\ \bot\end{array}}}{\varphi}\ \text{PBC} \qquad \frac{\varphi}{\neg\neg\varphi}\ \neg\neg i \qquad \frac{}{\varphi \vee \neg\varphi}\ \text{LEM}$$

Example

$((p \to q) \to p) \to p$ is valid:

1	$(p \to q) \to p$	assumption
10	p	
11	$((p \to q) \to p) \to p$	$\to$i 1–10

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3	p	assumption
4	$\neg p$	assumption
10	p	$\vee$e 2, 3–3, 4–9
11	$((p \to q) \to p) \to p$	$\to$i 1–10

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5	p	assumption
6	$\bot$	$\neg$e 5, 4
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7	q	$\bot$e 6
10	p	$\vee$e 2, 3–3, 4–9
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4	$\neg p$	assumption
5	p	assumption
6	$\bot$	$\neg$e 5, 4
7	q	$\bot$e 6
8	$p \to q$	$\to$i 5 – 7
10	p	$\vee$e 2, 3 – 3, 4 – 9
11	$((p \to q) \to p) \to p$	$\to$i 1 – 10

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5	p	assumption
6	$\bot$	$\neg$e 5, 4
7	q	$\bot$e 6
8	$p \to q$	$\to$i 5–7
9	p	$\to$e 1, 8
10	p	$\vee$e 2, 3–3, 4–9
11	$((p \to q) \to p) \to p$	$\to$i 1–10

Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, soundness and completeness, syntax, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching-time temporal logic, CTL*, fairness, linear-time temporal logic, model checking algorithms, symbolic model checking

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Theorem

natural deduction is complete:

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi \quad \Longrightarrow \quad \varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi \text{ is valid}$$

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all true statements can be proved

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Proof structure

① $\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi$ assumption

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Proof structure

① $\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi$ — assumption

② $\vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$

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② $\vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ — ① $\Longrightarrow$ ② easy

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③ $\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ is valid

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② $\vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ ① $\Longrightarrow$ ② easy

③ $\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ is valid ② $\Longrightarrow$ ③ difficult

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all true statements can be proved

Proof structure

①	$\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi$		assumption
②	$\vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$	① $\Longrightarrow$ ②	easy
③	$\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ is valid	② $\Longrightarrow$ ③	difficult
④	$\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is valid		

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Proof structure

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②	$\vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$	① $\Longrightarrow$ ②	easy
③	$\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ is valid	② $\Longrightarrow$ ③	difficult
④	$\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is valid	③ $\Longrightarrow$ ④	easy

Lemma ❶ $\Longrightarrow$ ❷

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi \quad \Longrightarrow \quad \vDash \varphi_1 \to (\varphi_2 \to (\cdots (\varphi_n \to \psi) \cdots))$$

Lemma ❶ $\Longrightarrow$ ❷

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi \quad \Longrightarrow \quad \vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$$

Proof

- suppose $\vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ does not hold

Lemma ❶ $\Longrightarrow$ ❷

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi \quad \Longrightarrow \quad \vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$$

Proof

- suppose $\vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ does not hold
- $\bar{v}(\varphi_1) = \cdots = \bar{v}(\varphi_n) = \mathsf{T}$ and $\bar{v}(\psi) = \mathsf{F}$ for some valuation v

Lemma ① $\Longrightarrow$ ②

$$\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi \quad \Longrightarrow \quad \vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$$

Proof

- suppose $\vDash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ does not hold
- $\bar{v}(\varphi_1) = \cdots = \bar{v}(\varphi_n) = \mathsf{T}$ and $\bar{v}(\psi) = \mathsf{F}$ for some valuation v
- $\varphi_1, \varphi_2, \ldots, \varphi_n \vDash \psi$ does not hold

Lemma ❸ $\Longrightarrow$ ❹

$\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ is valid $\Longrightarrow$ $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is valid

Lemma ❸ $\Longrightarrow$ ❹

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Proof

- Π: proof of validity of $\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$

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$\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ is valid $\Longrightarrow$ $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is valid

Proof

- Π: proof of validity of $\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$
- proof of validity of $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$:

$\varphi_1 \; \varphi_2 \; \cdots \; \varphi_n$ premises

Lemma ❸ $\Longrightarrow$ ❹

$\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ is valid $\quad\Longrightarrow\quad$ $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is valid

Proof

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- proof of validity of $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$:

$$\varphi_1 \; \varphi_2 \; \cdots \; \varphi_n \qquad \text{premises}$$

$$\Pi$$

$$\varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$$

Lemma ③ $\Longrightarrow$ ④

$\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$ is valid $\quad\Longrightarrow\quad \varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is valid

Proof

- Π: proof of validity of $\vdash \varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$
- proof of validity of $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$:

$\varphi_1 \; \varphi_2 \; \cdots \; \varphi_n$	premises
Π	
$\varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$	
$\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots)$	$\to$e

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$\varphi_1 \; \varphi_2 \; \cdots \; \varphi_n$	premises
Π	
$\varphi_1 \to (\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots))$	
$\varphi_2 \to (\cdots(\varphi_n \to \psi)\cdots)$	$\to$e
$\vdots$	$\vdots$
ψ	$\to$e

Lemma ❷ $\Longrightarrow$ ❸

$\vDash \varphi \quad \Longrightarrow \quad \vdash \varphi$ is valid

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$\vDash \varphi \quad \Longrightarrow \quad \vdash \varphi$ is valid

Definition

valuation v, formula φ

$$\langle\varphi\rangle^v = \begin{cases} \varphi & \text{if } \bar{v}(\varphi) = \mathsf{T} \\ \neg\varphi & \text{if } \bar{v}(\varphi) = \mathsf{F} \end{cases}$$

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Main Lemma

$p_1, \dots, p_n$ are all atoms in $\varphi \quad \Longrightarrow \quad \langle p_1\rangle^v, \dots, \langle p_n\rangle^v \vdash \langle\varphi\rangle^v$ is valid

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every line in truth table corresponds to valid sequent

Main Lemma

$p_1, \ldots, p_n$ are all atoms in $\varphi \implies \langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$ is valid

Example

formula $\varphi = \neg p \vee q$

valuation	p	q	
v_1	T	T	

Main Lemma

$p_1, \ldots, p_n$ are all atoms in $\varphi \implies \langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$ is valid

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formula $\varphi = \neg p \vee q$

valuation	p	q	sequent
v_1	T	T	$p, q \vdash \neg p \vee q$

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Example

formula $\varphi = \neg p \vee q$

valuation	p	q	sequent
v_1	T	T	$p, q \vdash \neg p \vee q$
v_2	T	F	$p, \neg q \vdash \neg(\neg p \vee q)$

Main Lemma

$p_1, \ldots, p_n$ are all atoms in $\varphi \implies \langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$ is valid

Example

formula $\varphi = \neg p \vee q$

valuation	p	q	sequent
v_1	T	T	$p, q \vdash \neg p \vee q$
v_2	T	F	$p, \neg q \vdash \neg(\neg p \vee q)$
v_3	F	T	$\neg p, q \vdash \neg p \vee q$

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$p_1, \ldots, p_n$ are all atoms in $\varphi \quad \Longrightarrow \quad \langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$ is valid

Example

formula $\varphi = \neg p \vee q$

valuation	p	q	sequent
v_1	T	T	$p, q \vdash \neg p \vee q$
v_2	T	F	$p, \neg q \vdash \neg(\neg p \vee q)$
v_3	F	T	$\neg p, q \vdash \neg p \vee q$
v_4	F	F	$\neg p, \neg q \vdash \neg p \vee q$

Main Lemma

$p_1, \ldots, p_n$ are all atoms in $\varphi \quad \Longrightarrow \quad \langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$ is valid

Proof

induction on structure of φ

Example

formula $\varphi = \neg p \vee q$

valuation	p	q	sequent
v_1	T	T	$p, q \vdash \neg p \vee q$
v_2	T	F	$p, \neg q \vdash \neg(\neg p \vee q)$
v_3	F	T	$\neg p, q \vdash \neg p \vee q$
v_4	F	F	$\neg p, \neg q \vdash \neg p \vee q$

Main Lemma

$p_1, \ldots, p_n$ are all atoms in $\varphi \implies \langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$ is valid

Proof (cont'd on subsequent slides)

induction on structure of φ

Example

formula $\varphi = \neg p \vee q$

valuation	p	q	sequent
v_1	T	T	$p, q \vdash \neg p \vee q$
v_2	T	F	$p, \neg q \vdash \neg(\neg p \vee q)$
v_3	F	T	$\neg p, q \vdash \neg p \vee q$
v_4	F	F	$\neg p, \neg q \vdash \neg p \vee q$

Base Case

$\varphi = p$

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- $v(p) = \mathsf{T}$: $\langle p \rangle^v = \langle \varphi \rangle^v = p$ $p \vdash p$ is valid

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Base Cases

$\varphi = p$

- $v(p) = \mathsf{T}$: $\langle p \rangle^v = \langle \varphi \rangle^v = p$ $\quad p \vdash p$ is valid
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$\varphi = \top$ $\quad \vdash \top$ is valid

Base Cases

$\varphi = p$

- $v(p) = \mathsf{T}$: $\langle p \rangle^v = \langle \varphi \rangle^v = p$ $p \vdash p$ is valid
- $v(p) = \mathsf{F}$: $\langle p \rangle^v = \langle \varphi \rangle^v = \neg p$ $\neg p \vdash \neg p$ is valid

$\varphi = \top$ $\vdash \top$ is valid

$\varphi = \bot$ $\vdash \neg\bot$ is valid

Base Cases

$\varphi = p$

- $v(p) = \mathsf{T}$: $\langle p \rangle^v = \langle \varphi \rangle^v = p$ $\quad p \vdash p$ is valid
- $v(p) = \mathsf{F}$: $\langle p \rangle^v = \langle \varphi \rangle^v = \neg p$ $\quad \neg p \vdash \neg p$ is valid

$\varphi = \top$ $\quad \vdash \top$ is valid

$\varphi = \bot$ $\quad \vdash \neg\bot$ is valid

Induction Step (4 cases)

case 1: $\varphi = \neg\psi$

Base Cases

$\varphi = p$

- $v(p) = \mathsf{T}$: $\langle p \rangle^v = \langle \varphi \rangle^v = p$ $\quad p \vdash p$ is valid
- $v(p) = \mathsf{F}$: $\langle p \rangle^v = \langle \varphi \rangle^v = \neg p$ $\quad \neg p \vdash \neg p$ is valid

$\varphi = \top$ $\quad \vdash \top$ is valid

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Induction Step (4 cases)

case 1: $\varphi = \neg\psi$

induction hypothesis: $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi \rangle^v$ is valid — Π

Base Cases

$\varphi = p$

- $v(p) = \text{T}$: $\langle p \rangle^v = \langle \varphi \rangle^v = p$ $\quad p \vdash p$ is valid
- $v(p) = \text{F}$: $\langle p \rangle^v = \langle \varphi \rangle^v = \neg p$ $\quad \neg p \vdash \neg p$ is valid

$\varphi = \top$ $\quad \vdash \top$ is valid

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Induction Step (4 cases)

case 1: $\varphi = \neg\psi$

induction hypothesis: $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi \rangle^v$ is valid — Π

- $\bar{v}(\varphi) = \text{T}$: $\langle \varphi \rangle^v = \varphi = \neg\psi = \langle \psi \rangle^v$

Base Cases

$\varphi = p$

- $v(p) = \mathrm{T}$: $\langle p \rangle^v = \langle \varphi \rangle^v = p$ $\quad p \vdash p$ is valid
- $v(p) = \mathrm{F}$: $\langle p \rangle^v = \langle \varphi \rangle^v = \neg p$ $\quad \neg p \vdash \neg p$ is valid

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Induction Step (4 cases)

case 1: $\varphi = \neg\psi$

induction hypothesis: $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi \rangle^v$ is valid — Π

- $\bar{v}(\varphi) = \mathrm{T}$: $\langle \varphi \rangle^v = \varphi = \neg\psi = \langle \psi \rangle^v$
- $\bar{v}(\varphi) = \mathrm{F}$: $\langle \varphi \rangle^v = \neg\varphi = \neg\neg\psi$ and $\langle \psi \rangle^v = \psi$

  extend Π with $\neg\neg$i to get proof of $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$

Induction Step (4 cases)

case 2: $\varphi = \psi_1 \wedge \psi_2$

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- $q_1, \ldots, q_l$: all atoms in ψ_1 $\qquad$ $r_1, \ldots, r_k$: all atoms in ψ_2

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- induction hypothesis: $\langle q_1 \rangle^v, \ldots, \langle q_l \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle r_1 \rangle^v, \ldots, \langle r_k \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid

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- $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — $\sqcap$

Induction Step (4 cases)

case 2: $\varphi = \psi_1 \wedge \psi_2$

- $q_1, \ldots, q_l$: all atoms in ψ_1 $\qquad r_1, \ldots, r_k$: all atoms in ψ_2
- induction hypothesis: $\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid
- $\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v \vdash \langle \varphi \rangle^v$ is valid

Induction Step (4 cases)

case 2: $\varphi = \psi_1 \wedge \psi_2$

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- $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v \vdash \langle \varphi \rangle^v$ is valid

$\bar{v}(\psi_1)$	$\bar{v}(\psi_2)$	$\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$	$\langle \varphi \rangle^v$
T	T	$\psi_1 \wedge \psi_2$	$\psi_1 \wedge \psi_2$
T	F	$\psi_1 \wedge \neg\psi_2$	$\neg(\psi_1 \wedge \psi_2)$
F	T	$\neg\psi_1 \wedge \psi_2$	$\neg(\psi_1 \wedge \psi_2)$
F	F	$\neg\psi_1 \wedge \neg\psi_2$	$\neg(\psi_1 \wedge \psi_2)$

Induction Step (4 cases)

case 2: $\varphi = \psi_1 \wedge \psi_2$

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- induction hypothesis: $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid
- $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v \vdash \langle \varphi \rangle^v$ is valid — Π'

$\bar{v}(\psi_1)$	$\bar{v}(\psi_2)$	$\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$	$\langle \varphi \rangle^v$
T	T	$\psi_1 \wedge \psi_2$	$\psi_1 \wedge \psi_2$
T	F	$\psi_1 \wedge \neg\psi_2$	$\neg(\psi_1 \wedge \psi_2)$
F	T	$\neg\psi_1 \wedge \psi_2$	$\neg(\psi_1 \wedge \psi_2)$
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- $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v \vdash \langle \varphi \rangle^v$ is valid — Π'

$\bar{v}(\psi_1)$	$\bar{v}(\psi_2)$	$\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$	$\langle \varphi \rangle^v$
T	T	$\psi_1 \wedge \psi_2$	$\psi_1 \wedge \psi_2$
T	F	$\psi_1 \wedge \neg\psi_2$	$\neg(\psi_1 \wedge \psi_2)$
F	T	$\neg\psi_1 \wedge \psi_2$	$\neg(\psi_1 \wedge \psi_2)$
F	F	$\neg\psi_1 \wedge \neg\psi_2$	$\neg(\psi_1 \wedge \psi_2)$

- combining Π and Π' yields validity of $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$

Induction Step (4 cases)

case 3: $\varphi = \psi_1 \vee \psi_2$

- $q_1, \dots, q_l$: all atoms in ψ_1 $r_1, \dots, r_k$: all atoms in ψ_2
- induction hypothesis: $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid
- $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v \vdash \langle \varphi \rangle^v$ is valid — Π'

$\bar{v}(\psi_1)$	$\bar{v}(\psi_2)$	$\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$	$\langle \varphi \rangle^v$
T	T	$\psi_1 \wedge \psi_2$	
T	F	$\psi_1 \wedge \neg\psi_2$	
F	T	$\neg\psi_1 \wedge \psi_2$	
F	F	$\neg\psi_1 \wedge \neg\psi_2$	

Induction Step (4 cases)

case 3: $\varphi = \psi_1 \vee \psi_2$

- $q_1, \dots, q_l$: all atoms in ψ_1 $r_1, \dots, r_k$: all atoms in ψ_2
- induction hypothesis: $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid
- $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v \vdash \langle \varphi \rangle^v$ is valid — Π'

$\bar{v}(\psi_1)$	$\bar{v}(\psi_2)$	$\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$	$\langle \varphi \rangle^v$
T	T	$\psi_1 \wedge \psi_2$	$\psi_1 \vee \psi_2$
T	F	$\psi_1 \wedge \neg\psi_2$	$\psi_1 \vee \psi_2$
F	T	$\neg\psi_1 \wedge \psi_2$	$\psi_1 \vee \psi_2$
F	F	$\neg\psi_1 \wedge \neg\psi_2$	$\neg(\psi_1 \vee \psi_2)$

Induction Step (4 cases)

case 3: $\varphi = \psi_1 \vee \psi_2$

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- induction hypothesis: $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid
- $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v \vdash \langle \varphi \rangle^v$ is valid — Π'

$\bar{v}(\psi_1)$	$\bar{v}(\psi_2)$	$\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$	$\langle \varphi \rangle^v$
T	T	$\psi_1 \wedge \psi_2$	$\psi_1 \vee \psi_2$
T	F	$\psi_1 \wedge \neg\psi_2$	$\psi_1 \vee \psi_2$
F	T	$\neg\psi_1 \wedge \psi_2$	$\psi_1 \vee \psi_2$
F	F	$\neg\psi_1 \wedge \neg\psi_2$	$\neg(\psi_1 \vee \psi_2)$

- combining Π and Π' yields validity of $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$

Induction Step (4 cases)

case 4: $\varphi = \psi_1 \to \psi_2$

- $q_1, \dots, q_l$: all atoms in ψ_1 $r_1, \dots, r_k$: all atoms in ψ_2
- induction hypothesis: $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid
- $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v \vdash \langle \varphi \rangle^v$ is valid — Π'

$\bar{v}(\psi_1)$	$\bar{v}(\psi_2)$	$\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$	$\langle \varphi \rangle^v$
T	T	$\psi_1 \wedge \psi_2$	
T	F	$\psi_1 \wedge \neg \psi_2$	
F	T	$\neg \psi_1 \wedge \psi_2$	
F	F	$\neg \psi_1 \wedge \neg \psi_2$	

Induction Step (4 cases)

case 4: $\varphi = \psi_1 \to \psi_2$

- $q_1, \ldots, q_l$: all atoms in ψ_1 $r_1, \ldots, r_k$: all atoms in ψ_2
- induction hypothesis: $\langle p_1\rangle^v, \ldots, \langle p_n\rangle^v \vdash \langle\psi_1\rangle^v$ and $\langle p_1\rangle^v, \ldots, \langle p_n\rangle^v \vdash \langle\psi_2\rangle^v$ are valid
- $\langle p_1\rangle^v, \ldots, \langle p_n\rangle^v \vdash \langle\psi_1\rangle^v \wedge \langle\psi_2\rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle\psi_1\rangle^v \wedge \langle\psi_2\rangle^v \vdash \langle\varphi\rangle^v$ is valid — Π'

$\bar{v}(\psi_1)$	$\bar{v}(\psi_2)$	$\langle\psi_1\rangle^v \wedge \langle\psi_2\rangle^v$	$\langle\varphi\rangle^v$
T	T	$\psi_1 \wedge \psi_2$	$\psi_1 \to \psi_2$
T	F	$\psi_1 \wedge \neg\psi_2$	$\neg(\psi_1 \to \psi_2)$
F	T	$\neg\psi_1 \wedge \psi_2$	$\psi_1 \to \psi_2$
F	F	$\neg\psi_1 \wedge \neg\psi_2$	$\psi_1 \to \psi_2$

Induction Step (4 cases)

case 4: $\varphi = \psi_1 \to \psi_2$

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- induction hypothesis: $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v$ and $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_2 \rangle^v$ are valid
- $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$ is valid (using $\wedge$i) — Π
- to prove: $\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v \vdash \langle \varphi \rangle^v$ is valid — Π'

$\bar{v}(\psi_1)$	$\bar{v}(\psi_2)$	$\langle \psi_1 \rangle^v \wedge \langle \psi_2 \rangle^v$	$\langle \varphi \rangle^v$
T	T	$\psi_1 \wedge \psi_2$	$\psi_1 \to \psi_2$
T	F	$\psi_1 \wedge \neg\psi_2$	$\neg(\psi_1 \to \psi_2)$
F	T	$\neg\psi_1 \wedge \psi_2$	$\psi_1 \to \psi_2$
F	F	$\neg\psi_1 \wedge \neg\psi_2$	$\psi_1 \to \psi_2$

- combining Π and Π' yields validity of $\langle p_1 \rangle^v, \dots, \langle p_n \rangle^v \vdash \langle \varphi \rangle^v$

Theorem

$\vDash \varphi \implies \vdash \varphi$ is valid

Proof

suppose $\vDash \varphi$

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- for every valuation v $\langle\varphi\rangle^v = \varphi$

Theorem

$\vDash \varphi \implies \vdash \varphi$ is valid

Proof

suppose $\vDash \varphi$

- for every valuation v $\langle \varphi \rangle^v = \varphi$
- for every valuation v $\langle p_1 \rangle^v, \ldots, \langle p_n \rangle^v \vdash \varphi$ is valid sequent

Theorem

$\models \varphi \implies \vdash \varphi$ is valid

Proof

suppose $\models \varphi$

- for every valuation v $\langle\varphi\rangle^v = \varphi$
- for every valuation v $\langle p_1\rangle^v, \dots, \langle p_n\rangle^v \vdash \varphi$ is valid sequent
- combine all proofs of these sequents into proof of validity of

$$\vdash \varphi$$

by $2^n - 1$ applications of LEM and $\vee$e

Example

$\vdash p \land q \to q$ is valid

Example

$\vdash p \land q \rightarrow q$ is valid

valuation	p	q	
v_1	T	T	

Example

$\vdash p \wedge q \rightarrow q$ is valid

valuation	p	q	sequent
v_1	T	T	$p, q \vdash p \wedge q \rightarrow q$

Example

$\vdash p \land q \to q$ is valid

valuation	p	q	sequent	proof
v_1	T	T	$p, q \vdash p \land q \to q$	Π_1

Example

$\vdash p \wedge q \rightarrow q$ is valid

valuation	p	q	sequent	proof
v_1	T	T	$p, q \vdash p \wedge q \rightarrow q$	Π_1
v_2	T	F	$p, \neg q \vdash p \wedge q \rightarrow q$	Π_2

Example

$\vdash p \land q \to q$ is valid

valuation	p	q	sequent	proof
v_1	T	T	$p, q \vdash p \land q \to q$	Π_1
v_2	T	F	$p, \neg q \vdash p \land q \to q$	Π_2
v_3	F	T	$\neg p, q \vdash p \land q \to q$	Π_3

Example

$\vdash p \land q \to q$ is valid

valuation	p	q	sequent	proof
v_1	T	T	$p, q \vdash p \land q \to q$	Π_1
v_2	T	F	$p, \neg q \vdash p \land q \to q$	Π_2
v_3	F	T	$\neg p, q \vdash p \land q \to q$	Π_3
v_4	F	F	$\neg p, \neg q \vdash p \land q \to q$	Π_4

Example

$\vdash p \wedge q \rightarrow q$ is valid

valuation	p	q	sequent	proof
v_1	T	T	$p, q \vdash p \wedge q \rightarrow q$	Π_1
v_2	T	F	$p, \neg q \vdash p \wedge q \rightarrow q$	Π_2
v_3	F	T	$\neg p, q \vdash p \wedge q \rightarrow q$	Π_3
v_4	F	F	$\neg p, \neg q \vdash p \wedge q \rightarrow q$	Π_4

$p \vee \neg p$ LEM

> p assumption
>
> $q \vee \neg q$ LEM
>
> > q ass
> >
> > $\cdots \Pi_1 \cdots$
> >
> > $p \wedge q \rightarrow q$
>
> > $\neg q$ ass
> >
> > $\cdots \Pi_2 \cdots$
> >
> > $p \wedge q \rightarrow q$
>
> $p \wedge q \rightarrow q$ $\vee$e

> $\neg p$ assumption
>
> $q \vee \neg q$ LEM
>
> > q ass
> >
> > $\cdots \Pi_3 \cdots$
> >
> > $p \wedge q \rightarrow q$
>
> > $\neg q$ ass
> >
> > $\cdots \Pi_4 \cdots$
> >
> > $p \wedge q \rightarrow q$
>
> $p \wedge q \rightarrow q$ $\vee$e

$p \wedge q \rightarrow q$ $\vee$e

Demo

Natural Deduction Tool

by Andreas Schnabl (2005)

Outline

Definitions

- clause is set of literals $\{\ell_1, \dots, \ell_n\}$ representing formula

$$\begin{cases} (\ell_1 \vee \cdots \vee \ell_n) & \text{if } n \geqslant 1 \\ \end{cases}$$

Definitions

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$$\begin{cases} (\ell_1 \vee \cdots \vee \ell_n) & \text{if } n \geqslant 1 \\ \bot & \text{if } n = 0 \end{cases}$$

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- $\Box$ denotes empty clause $\varnothing$

Definitions

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- $\square$ denotes empty clause $\varnothing$
- clausal form is set of clauses $\{C_1, \dots, C_m\}$ representing formula

$$\begin{cases} C_1 \wedge \cdots \wedge C_m & \text{if } m \geqslant 1 \\ \top & \text{if } m = 0 \end{cases}$$

Definitions

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- $\square$ denotes empty clause $\varnothing$
- clausal form is set of clauses $\{C_1, \ldots, C_m\}$ representing formula

$$\begin{cases} C_1 \wedge \cdots \wedge C_m & \text{if } m \geqslant 1 \\ \top & \text{if } m = 0 \end{cases}$$

Remark

every CNF can be written in clausal form

Example

CNF	clausal form
$(\neg p) \wedge (\neg q \vee \neg p) \wedge (\neg p \vee \neg r)$	

Example

CNF	clausal form
$(\neg p) \wedge (\neg q \vee \neg p) \wedge (\neg p \vee \neg r)$	$\{\{\neg p\}, \{\neg q, \neg p\}, \{\neg p, \neg r\}\}$

Example

CNF	clausal form
$(\neg p) \wedge (\neg q \vee \neg p) \wedge (\neg p \vee \neg r)$	$\{\{\neg p\}, \{\neg q, \neg p\}, \{\neg p, \neg r\}\}$
$(\neg p \vee q) \wedge (q \vee \neg r) \wedge (p \vee q \vee \neg r)$	

Example

CNF	clausal form
$(\neg p) \wedge (\neg q \vee \neg p) \wedge (\neg p \vee \neg r)$	$\{\{\neg p\}, \{\neg q, \neg p\}, \{\neg p, \neg r\}\}$
$(\neg p \vee q) \wedge (q \vee \neg r) \wedge (p \vee q \vee \neg r)$	$\{\{\neg p, q\}, \{q, \neg r\}, \{p, q, \neg r\}\}$

Example

CNF	clausal form
$(\neg p) \wedge (\neg q \vee \neg p) \wedge (\neg p \vee \neg r)$	$\{\{\neg p\}, \{\neg q, \neg p\}, \{\neg p, \neg r\}\}$
$(\neg p \vee q) \wedge (q \vee \neg r) \wedge (p \vee q \vee \neg r)$	$\{\{\neg p, q\}, \{q, \neg r\}, \{p, q, \neg r\}\}$
$(\neg p \vee \neg p) \wedge (q \vee r) \wedge (r \vee q)$	

Example

CNF	clausal form
$(\neg p) \wedge (\neg q \vee \neg p) \wedge (\neg p \vee \neg r)$	$\{\{\neg p\}, \{\neg q, \neg p\}, \{\neg p, \neg r\}\}$
$(\neg p \vee q) \wedge (q \vee \neg r) \wedge (p \vee q \vee \neg r)$	$\{\{\neg p, q\}, \{q, \neg r\}, \{p, q, \neg r\}\}$
$(\neg p \vee \neg p) \wedge (q \vee r) \wedge (r \vee q)$	$\{\{\neg p\}, \{q, r\}\}$

Example

CNF	clausal form
$(\neg p) \wedge (\neg q \vee \neg p) \wedge (\neg p \vee \neg r)$	$\{\{\neg p\}, \{\neg q, \neg p\}, \{\neg p, \neg r\}\}$
$(\neg p \vee q) \wedge (q \vee \neg r) \wedge (p \vee q \vee \neg r)$	$\{\{\neg p, q\}, \{q, \neg r\}, \{p, q, \neg r\}\}$
$(\neg p \vee \neg p) \wedge (q \vee r) \wedge (r \vee q)$	$\{\{\neg p\}, \{q, r\}\}$

Definition

literals ℓ_1 and ℓ_2 are complementary if $\underbrace{\ell_1 = \neg \ell_2 \text{ or } \neg \ell_1 = \ell_2}_{\ell_1 = \ell_2^c}$

Example

CNF	clausal form
$(\neg p) \wedge (\neg q \vee \neg p) \wedge (\neg p \vee \neg r)$	$\{\{\neg p\}, \{\neg q, \neg p\}, \{\neg p, \neg r\}\}$
$(\neg p \vee q) \wedge (q \vee \neg r) \wedge (p \vee q \vee \neg r)$	$\{\{\neg p, q\}, \{q, \neg r\}, \{p, q, \neg r\}\}$
$(\neg p \vee \neg p) \wedge (q \vee r) \wedge (r \vee q)$	$\{\{\neg p\}, \{q, r\}\}$

Definition

literals ℓ_1 and ℓ_2 are complementary if $\underbrace{\ell_1 = \neg \ell_2 \text{ or } \neg \ell_1 = \ell_2}_{\ell_1 = \ell_2^c}$

Notation

if ℓ is literal then $\ell^c = \begin{cases} \neg p & \text{if } \ell = p \\ p & \text{if } \ell = \neg p \end{cases}$

Definition

- clauses C_1 and C_2 clash on literal ℓ if $\ell \in C_1$ and $\ell^c \in C_2$

Definition

- clauses C_1 and C_2 clash on literal ℓ if $\ell \in C_1$ and $\ell^c \in C_2$
- resolvent of clauses C_1 and C_2 clashing on literal ℓ is clause

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- clauses $\{\neg p, \neg q, r\}$ and $\{p, r\}$ clash

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Examples

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Examples

- clauses $\{\neg p, \neg q, r\}$ and $\{p, r\}$ clash with resolvent $\{\neg q, r\}$
- clauses $\{p, \neg p, q\}$ and $\{q, r\}$ do not clash

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- clauses $\{\neg p, \neg q\}$ and $\{p, q\}$ clash with resolvents $\{\neg q, q\}$ and $\{\neg p, p\}$

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- resolvent of clauses C_1 and C_2 clashing on literal ℓ is clause

$$(C_1 \setminus \{\ell\}) \cup (C_2 \setminus \{\ell^c\})$$

- C_1 and C_2 are parent clauses of resolvent

Examples

- clauses $\{\neg p, \neg q, r\}$ and $\{p, r\}$ clash with resolvent $\{\neg q, r\}$
- clauses $\{p, \neg p, q\}$ and $\{q, r\}$ do not clash
- clauses $\{\neg p, \neg q\}$ and $\{p, q\}$ clash with resolvents $\{\neg q, q\}$ and $\{\neg p, p\}$
- clauses $\{p, \neg p, q\}$ and $\{p, \neg p, q\}$ clash with resolvent $\{p, \neg p, q\}$

Resolution

input: clausal form S

output: yes if S is satisfiable

no if S is unsatisfiable

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Theorem

resolution is terminating

Definition

refutation of S is resolution derivation of $\Box$ from S

Definition

refutation of S is resolution derivation of $\square$ from S

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resolution is sound and complete

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resolution is sound and complete:

$$S \text{ admits refutation} \iff \text{clausal form } S \text{ is unsatisfiable}$$

Definition

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Theorem

resolution is sound and complete:

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Definition

- resolvent of clauses C_1 and C_2 clashing on literal ℓ is clause $(C_1 \setminus \{\ell\}) \cup (C_2 \setminus \{\ell^c\})$
- special case (unit resolution): $C_1 = \{\ell\}$ with resolvent $C_2 \setminus \{\ell^c\}$

Example ❶

$$(\neg p \vee \neg q \vee r) \wedge (p \vee r) \wedge (q \vee r) \wedge \neg r$$

1 $\{\neg p, \neg q, r\}$

2 $\{p, r\}$

3 $\{q, r\}$

4 $\{\neg r\}$

Example ①

$$(\neg p \vee \neg q \vee r) \wedge (p \vee r) \wedge (q \vee r) \wedge \neg r$$

1 $\{\neg p, \neg q, r\}$

2 $\{p, r\}$

3 $\{q, r\}$

4 $\{\neg r\}$

Example ❶

$$(\neg p \vee \neg q \vee r) \wedge (p \vee r) \wedge (q \vee r) \wedge \neg r$$

1 $\{\neg p, \neg q, r\}$

2 $\{p, r\}$

3 $\{q, r\}$

4 $\{\neg r\}$

5 $\{\neg q, r\}$ resolve 1, 2, p

Example ❶

$$(\neg p \vee \neg q \vee r) \wedge (p \vee r) \wedge (q \vee r) \wedge \neg r$$

1 $\{\neg p, \neg q, r\}$

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3 $\{q, r\}$

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5 $\{\neg q, r\}$ resolve 1, 2, p

Example ❶

$$(\neg p \vee \neg q \vee r) \wedge (p \vee r) \wedge (q \vee r) \wedge \neg r$$

1 $\{\neg p, \neg q, r\}$

2 $\{p, r\}$

3 $\{q, r\}$

4 $\{\neg r\}$

5 $\{\neg q, r\}$ resolve 1, 2, p

6 $\{r\}$ resolve 3, 5, q

Example ❶

$$(\neg p \vee \neg q \vee r) \wedge (p \vee r) \wedge (q \vee r) \wedge \neg r$$

1 $\{\neg p, \neg q, r\}$

2 $\{p, r\}$

3 $\{q, r\}$

4 $\{\neg r\}$

5 $\{\neg q, r\}$ resolve 1, 2, p

6 $\{r\}$ resolve 3, 5, q

Example ➊

$$(\neg p \vee \neg q \vee r) \wedge (p \vee r) \wedge (q \vee r) \wedge \neg r$$

1	$\{\neg p, \neg q, r\}$	
2	$\{p, r\}$	
3	$\{q, r\}$	
4	$\{\neg r\}$	
5	$\{\neg q, r\}$	resolve 1, 2, p
6	$\{r\}$	resolve 3, 5, q
7	$\square$	resolve 4, 6, r

Example ❶

$$(\neg p \lor \neg q \lor r) \land (p \lor r) \land (q \lor r) \land \neg r$$

1	$\{\neg p, \neg q, r\}$	
2	$\{p, r\}$	
3	$\{q, r\}$	
4	$\{\neg r\}$	
5	$\{\neg q, r\}$	resolve 1, 2, p
6	$\{r\}$	resolve 3, 5, q
7	$\square$	resolve 4, 6, r

unsatisfiable

Example ❷

$$p \wedge (p \rightarrow ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \rightarrow ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \rightarrow q) \wedge (\neg r \rightarrow t) \wedge (t \rightarrow s)$$

1 $\{p\}$

2 $\{\neg p, q, r\}$

3 $\{\neg p, \neg q, \neg r\}$

4 $\{\neg p, s, t\}$

5 $\{\neg p, \neg s, \neg t\}$

6 $\{\neg s, q\}$

7 $\{r, t\}$

8 $\{\neg t, s\}$

Example ❷

$$p \wedge (p \rightarrow ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \rightarrow ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \rightarrow q) \wedge (\neg r \rightarrow t) \wedge (t \rightarrow s)$$

1 $\{p\}$

2 $\{\neg p, q, r\}$

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4 $\{\neg p, s, t\}$

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Example ❷

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1 $\{p\}$

2 $\{\neg p, q, r\}$

3 $\{\neg p, \neg q, \neg r\}$

4 $\{\neg p, s, t\}$

5 $\{\neg p, \neg s, \neg t\}$

6 $\{\neg s, q\}$

7 $\{r, t\}$

8 $\{\neg t, s\}$

9 $\{\neg q, \neg r\}$ resolve 1, 3, p

Example ❷

$$p \wedge (p \rightarrow ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \rightarrow ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \rightarrow q) \wedge (\neg r \rightarrow t) \wedge (t \rightarrow s)$$

1 $\{p\}$

2 $\{\neg p, q, r\}$

3 $\{\neg p, \neg q, \neg r\}$

4 $\{\neg p, s, t\}$

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1 $\{p\}$

2 $\{\neg p, q, r\}$

3 $\{\neg p, \neg q, \neg r\}$

4 $\{\neg p, s, t\}$

5 $\{\neg p, \neg s, \neg t\}$

6 $\{\neg s, q\}$

7 $\{r, t\}$

8 $\{\neg t, s\}$

9 $\{\neg q, \neg r\}$ resolve 1, 3, p

10 $\{s, t\}$ resolve 1, 4, p

Example ❷

$$p \wedge (p \rightarrow ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \rightarrow ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \rightarrow q) \wedge (\neg r \rightarrow t) \wedge (t \rightarrow s)$$

1 $\{p\}$
2 $\{\neg p, q, r\}$
3 $\{\neg p, \neg q, \neg r\}$
4 $\{\neg p, s, t\}$
5 $\{\neg p, \neg s, \neg t\}$
6 $\{\neg s, q\}$
7 $\{r, t\}$
8 $\{\neg t, s\}$
9 $\{\neg q, \neg r\}$ resolve 1, 3, p
10 $\{s, t\}$ resolve 1, 4, p

Example ❷

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1 $\{p\}$

2 $\{\neg p, q, r\}$

3 $\{\neg p, \neg q, \neg r\}$

4 $\{\neg p, s, t\}$

5 $\{\neg p, \neg s, \neg t\}$

6 $\{\neg s, q\}$

7 $\{r, t\}$

8 $\{\neg t, s\}$

9 $\{\neg q, \neg r\}$ resolve 1, 3, p

10 $\{s, t\}$ resolve 1, 4, p

11 $\{\neg s, \neg t\}$ resolve 1, 5, p

Example ❷

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1 $\{p\}$

2 $\{\neg p, q, r\}$

3 $\{\neg p, \neg q, \neg r\}$

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$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1	$\{p\}$	
2	$\{\neg p, q, r\}$	
3	$\{\neg p, \neg q, \neg r\}$	
4	$\{\neg p, s, t\}$	
5	$\{\neg p, \neg s, \neg t\}$	
6	$\{\neg s, q\}$	
7	$\{r, t\}$	
8	$\{\neg t, s\}$	
9	$\{\neg q, \neg r\}$	resolve 1, 3, p
10	$\{s, t\}$	resolve 1, 4, p
11	$\{\neg s, \neg t\}$	resolve 1, 5, p
12	$\{\neg s, \neg r\}$	resolve 6, 9, q

Example ❷

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1 $\{p\}$

2 $\{\neg p, q, r\}$

3 $\{\neg p, \neg q, \neg r\}$

4 $\{\neg p, s, t\}$

5 $\{\neg p, \neg s, \neg t\}$

6 $\{\neg s, q\}$

7 $\{r, t\}$

8 $\{\neg t, s\}$

9 $\{\neg q, \neg r\}$ resolve 1, 3, p

10 $\{s, t\}$ resolve 1, 4, p

11 $\{\neg s, \neg t\}$ resolve 1, 5, p

12 $\{\neg s, \neg r\}$ resolve 6, 9, q

Example ❷

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1	$\{p\}$		10	$\{s, t\}$	resolve 1, 4, p
2	$\{\neg p, q, r\}$		11	$\{\neg s, \neg t\}$	resolve 1, 5, p
3	$\{\neg p, \neg q, \neg r\}$		12	$\{\neg s, \neg r\}$	resolve 6, 9, q
4	$\{\neg p, s, t\}$		13	$\{s\}$	resolve 8, 10, t
5	$\{\neg p, \neg s, \neg t\}$				
6	$\{\neg s, q\}$				
7	$\{r, t\}$				
8	$\{\neg t, s\}$				
9	$\{\neg q, \neg r\}$	resolve 1, 3, p			

Example ②

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1	$\{p\}$	
2	$\{\neg p, q, r\}$	
3	$\{\neg p, \neg q, \neg r\}$	
4	$\{\neg p, s, t\}$	
5	$\{\neg p, \neg s, \neg t\}$	
6	$\{\neg s, q\}$	
7	$\{r, t\}$	
8	$\{\neg t, s\}$	
9	$\{\neg q, \neg r\}$	resolve 1, 3, p
10	$\{s, t\}$	resolve 1, 4, p
11	$\{\neg s, \neg t\}$	resolve 1, 5, p
12	$\{\neg s, \neg r\}$	resolve 6, 9, q
13	$\{s\}$	resolve 8, 10, t

Example ❷

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1	$\{p\}$	
2	$\{\neg p, q, r\}$	
3	$\{\neg p, \neg q, \neg r\}$	
4	$\{\neg p, s, t\}$	
5	$\{\neg p, \neg s, \neg t\}$	
6	$\{\neg s, q\}$	
7	$\{r, t\}$	
8	$\{\neg t, s\}$	
9	$\{\neg q, \neg r\}$	resolve 1, 3, p
10	$\{s, t\}$	resolve 1, 4, p
11	$\{\neg s, \neg t\}$	resolve 1, 5, p
12	$\{\neg s, \neg r\}$	resolve 6, 9, q
13	$\{s\}$	resolve 8, 10, t
14	$\{\neg t\}$	resolve 11, 13, s

Example ❷

$$p \wedge (p \rightarrow ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \rightarrow ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \rightarrow q) \wedge (\neg r \rightarrow t) \wedge (t \rightarrow s)$$

1	$\{p\}$	
2	$\{\neg p, q, r\}$	
3	$\{\neg p, \neg q, \neg r\}$	
4	$\{\neg p, s, t\}$	
5	$\{\neg p, \neg s, \neg t\}$	
6	$\{\neg s, q\}$	
7	$\{r, t\}$	
8	$\{\neg t, s\}$	
9	$\{\neg q, \neg r\}$	resolve 1, 3, p
10	$\{s, t\}$	resolve 1, 4, p
11	$\{\neg s, \neg t\}$	resolve 1, 5, p
12	$\{\neg s, \neg r\}$	resolve 6, 9, q
13	$\{s\}$	resolve 8, 10, t
14	$\{\neg t\}$	resolve 11, 13, s

Example ❷

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1	$\{p\}$		10	$\{s, t\}$	resolve 1, 4, p
2	$\{\neg p, q, r\}$		11	$\{\neg s, \neg t\}$	resolve 1, 5, p
3	$\{\neg p, \neg q, \neg r\}$		12	$\{\neg s, \neg r\}$	resolve 6, 9, q
4	$\{\neg p, s, t\}$		13	$\{s\}$	resolve 8, 10, t
5	$\{\neg p, \neg s, \neg t\}$		14	$\{\neg t\}$	resolve 11, 13, s
6	$\{\neg s, q\}$		15	$\{r\}$	resolve 7, 14, t
7	$\{r, t\}$				
8	$\{\neg t, s\}$				
9	$\{\neg q, \neg r\}$	resolve 1, 3, p			

Example ❷

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1	$\{p\}$	
2	$\{\neg p, q, r\}$	
3	$\{\neg p, \neg q, \neg r\}$	
4	$\{\neg p, s, t\}$	
5	$\{\neg p, \neg s, \neg t\}$	
6	$\{\neg s, q\}$	
7	$\{r, t\}$	
8	$\{\neg t, s\}$	
9	$\{\neg q, \neg r\}$	resolve 1, 3, p
10	$\{s, t\}$	resolve 1, 4, p
11	$\{\neg s, \neg t\}$	resolve 1, 5, p
12	$\{\neg s, \neg r\}$	resolve 6, 9, q
13	$\{s\}$	resolve 8, 10, t
14	$\{\neg t\}$	resolve 11, 13, s
15	$\{r\}$	resolve 7, 14, t

Example ❷

$$p \wedge (p \rightarrow ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \rightarrow ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \rightarrow q) \wedge (\neg r \rightarrow t) \wedge (t \rightarrow s)$$

1	$\{p\}$	
2	$\{\neg p, q, r\}$	
3	$\{\neg p, \neg q, \neg r\}$	
4	$\{\neg p, s, t\}$	
5	$\{\neg p, \neg s, \neg t\}$	
6	$\{\neg s, q\}$	
7	$\{r, t\}$	
8	$\{\neg t, s\}$	
9	$\{\neg q, \neg r\}$	resolve 1, 3, p
10	$\{s, t\}$	resolve 1, 4, p
11	$\{\neg s, \neg t\}$	resolve 1, 5, p
12	$\{\neg s, \neg r\}$	resolve 6, 9, q
13	$\{s\}$	resolve 8, 10, t
14	$\{\neg t\}$	resolve 11, 13, s
15	$\{r\}$	resolve 7, 14, t
16	$\{\neg r\}$	resolve 12, 13, s

Example ❷

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1	$\{p\}$	
2	$\{\neg p, q, r\}$	
3	$\{\neg p, \neg q, \neg r\}$	
4	$\{\neg p, s, t\}$	
5	$\{\neg p, \neg s, \neg t\}$	
6	$\{\neg s, q\}$	
7	$\{r, t\}$	
8	$\{\neg t, s\}$	
9	$\{\neg q, \neg r\}$	resolve 1, 3, p
10	$\{s, t\}$	resolve 1, 4, p
11	$\{\neg s, \neg t\}$	resolve 1, 5, p
12	$\{\neg s, \neg r\}$	resolve 6, 9, q
13	$\{s\}$	resolve 8, 10, t
14	$\{\neg t\}$	resolve 11, 13, s
15	$\{r\}$	resolve 7, 14, t
16	$\{\neg r\}$	resolve 12, 13, s

Example ❷

$$p \wedge (p \rightarrow ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \rightarrow ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \rightarrow q) \wedge (\neg r \rightarrow t) \wedge (t \rightarrow s)$$

1	$\{p\}$	
2	$\{\neg p, q, r\}$	
3	$\{\neg p, \neg q, \neg r\}$	
4	$\{\neg p, s, t\}$	
5	$\{\neg p, \neg s, \neg t\}$	
6	$\{\neg s, q\}$	
7	$\{r, t\}$	
8	$\{\neg t, s\}$	
9	$\{\neg q, \neg r\}$	resolve 1, 3, p
10	$\{s, t\}$	resolve 1, 4, p
11	$\{\neg s, \neg t\}$	resolve 1, 5, p
12	$\{\neg s, \neg r\}$	resolve 6, 9, q
13	$\{s\}$	resolve 8, 10, t
14	$\{\neg t\}$	resolve 11, 13, s
15	$\{r\}$	resolve 7, 14, t
16	$\{\neg r\}$	resolve 12, 13, s
17	$\square$	resolve 15, 16, r

Example ❷

$$p \wedge (p \to ((q \vee r) \wedge \neg(q \wedge r))) \wedge (p \to ((s \vee t) \wedge \neg(s \wedge t))) \wedge (s \to q) \wedge (\neg r \to t) \wedge (t \to s)$$

1	$\{p\}$	
2	$\{\neg p, q, r\}$	
3	$\{\neg p, \neg q, \neg r\}$	
4	$\{\neg p, s, t\}$	
5	$\{\neg p, \neg s, \neg t\}$	
6	$\{\neg s, q\}$	
7	$\{r, t\}$	
8	$\{\neg t, s\}$	
9	$\{\neg q, \neg r\}$	resolve 1, 3, p
10	$\{s, t\}$	resolve 1, 4, p
11	$\{\neg s, \neg t\}$	resolve 1, 5, p
12	$\{\neg s, \neg r\}$	resolve 6, 9, q
13	$\{s\}$	resolve 8, 10, t
14	$\{\neg t\}$	resolve 11, 13, s
15	$\{r\}$	resolve 7, 14, t
16	$\{\neg r\}$	resolve 12, 13, s
17	$\square$	resolve 15, 16, r

unsatisfiable

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1 $\{p, q\}$

2 $\{\neg p, \neg r\}$

3 $\{\neg q, s\}$

4 $\{p, \neg r\}$

5 $\{r, \neg s\}$

Example ③

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1 $\{p, q\}$

2 $\{\neg p, \neg r\}$

3 $\{\neg q, s\}$

4 $\{p, \neg r\}$

5 $\{r, \neg s\}$

6 $\{q, \neg r\}$ resolve 1, 2, p

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1 $\{p, q\}$

2 $\{\neg p, \neg r\}$

3 $\{\neg q, s\}$

4 $\{p, \neg r\}$

5 $\{r, \neg s\}$

6 $\{q, \neg r\}$ resolve 1, 2, p

7 $\{p, s\}$ resolve 1, 3, q

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1	$\{p, q\}$	
2	$\{\neg p, \neg r\}$	
3	$\{\neg q, s\}$	
4	$\{p, \neg r\}$	
5	$\{r, \neg s\}$	
6	$\{q, \neg r\}$	resolve 1, 2, p
7	$\{p, s\}$	resolve 1, 3, q
8	$\{\neg r\}$	resolve 2, 4, p

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1 $\{p, q\}$

2 $\{\neg p, \neg r\}$

3 $\{\neg q, s\}$

4 $\{p, \neg r\}$

5 $\{r, \neg s\}$

6 $\{q, \neg r\}$ resolve 1, 2, p

7 $\{p, s\}$ resolve 1, 3, q

8 $\{\neg r\}$ resolve 2, 4, p

9 $\{\neg p, \neg s\}$ resolve 2, 5, r

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1	$\{p, q\}$	
2	$\{\neg p, \neg r\}$	
3	$\{\neg q, s\}$	
4	$\{p, \neg r\}$	
5	$\{r, \neg s\}$	
6	$\{q, \neg r\}$	resolve 1, 2, p
7	$\{p, s\}$	resolve 1, 3, q
8	$\{\neg r\}$	resolve 2, 4, p
9	$\{\neg p, \neg s\}$	resolve 2, 5, r
10	$\{\neg r, s\}$	resolve 2, 7, p

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1	$\{p, q\}$	
2	$\{\neg p, \neg r\}$	
3	$\{\neg q, s\}$	
4	$\{p, \neg r\}$	
5	$\{r, \neg s\}$	
6	$\{q, \neg r\}$	resolve 1, 2, p
7	$\{p, s\}$	resolve 1, 3, q
8	$\{\neg r\}$	resolve 2, 4, p
9	$\{\neg p, \neg s\}$	resolve 2, 5, r
10	$\{\neg r, s\}$	resolve 2, 7, p
11	$\{\neg q, r\}$	resolve 3, 5, s

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1	$\{p, q\}$		12	$\{\neg p, \neg q\}$	resolve 3, 9, s	23	$\{q, \neg q\}$	resolve 6, 11, r
2	$\{\neg p, \neg r\}$		13	$\{p, \neg s\}$	resolve 4, 5, r	24	$\{p, \neg p\}$	resolve 7, 9, s
3	$\{\neg q, s\}$		14	$\{\neg r, \neg s\}$	resolve 4, 9, p	25	$\{p\}$	resolve 7, 19, s
4	$\{p, \neg r\}$		15	$\{p, \neg q\}$	resolve 4, 11, r	26	$\{\neg q\}$	resolve 8, 11, r
5	$\{r, \neg s\}$		16	$\{\neg q, \neg r\}$	resolve 4, 12, p			
6	$\{q, \neg r\}$	resolve 1, 2, p	17	$\{q, \neg s\}$	resolve 5, 6, r			
7	$\{p, s\}$	resolve 1, 3, q	18	$\{p, r\}$	resolve 5, 7, s			
8	$\{\neg r\}$	resolve 2, 4, p	19	$\{\neg s\}$	resolve 5, 8, r			
9	$\{\neg p, \neg s\}$	resolve 2, 5, r	20	$\{s, \neg s\}$	resolve 5, 10, r			
10	$\{\neg r, s\}$	resolve 2, 7, p	21	$\{r, \neg r\}$	resolve 5, 10, s			
11	$\{\neg q, r\}$	resolve 3, 5, s	22	$\{\neg q, \neg s\}$	resolve 5, 16, r			

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1	$\{p, q\}$	
2	$\{\neg p, \neg r\}$	
3	$\{\neg q, s\}$	
4	$\{p, \neg r\}$	
5	$\{r, \neg s\}$	
6	$\{q, \neg r\}$	resolve 1, 2, p
7	$\{p, s\}$	resolve 1, 3, q
8	$\{\neg r\}$	resolve 2, 4, p
9	$\{\neg p, \neg s\}$	resolve 2, 5, r
10	$\{\neg r, s\}$	resolve 2, 7, p
11	$\{\neg q, r\}$	resolve 3, 5, s
12	$\{\neg p, \neg q\}$	resolve 3, 9, s
13	$\{p, \neg s\}$	resolve 4, 5, r
14	$\{\neg r, \neg s\}$	resolve 4, 9, p
15	$\{p, \neg q\}$	resolve 4, 11, r
16	$\{\neg q, \neg r\}$	resolve 4, 12, p
17	$\{q, \neg s\}$	resolve 5, 6, r
18	$\{p, r\}$	resolve 5, 7, s
19	$\{\neg s\}$	resolve 5, 8, r
20	$\{s, \neg s\}$	resolve 5, 10, r
21	$\{r, \neg r\}$	resolve 5, 10, s
22	$\{\neg q, \neg s\}$	resolve 5, 16, r
23	$\{q, \neg q\}$	resolve 6, 11, r
24	$\{p, \neg p\}$	resolve 7, 9, s
25	$\{p\}$	resolve 7, 19, s
26	$\{\neg q\}$	resolve 8, 11, r

no further resolvents

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1	$\{p, q\}$	
2	$\{\neg p, \neg r\}$	
3	$\{\neg q, s\}$	
4	$\{p, \neg r\}$	
5	$\{r, \neg s\}$	
6	$\{q, \neg r\}$	resolve 1, 2, p
7	$\{p, s\}$	resolve 1, 3, q
8	$\{\neg r\}$	resolve 2, 4, p
9	$\{\neg p, \neg s\}$	resolve 2, 5, r
10	$\{\neg r, s\}$	resolve 2, 7, p
11	$\{\neg q, r\}$	resolve 3, 5, s
12	$\{\neg p, \neg q\}$	resolve 3, 9, s
13	$\{p, \neg s\}$	resolve 4, 5, r
14	$\{\neg r, \neg s\}$	resolve 4, 9, p
15	$\{p, \neg q\}$	resolve 4, 11, r
16	$\{\neg q, \neg r\}$	resolve 4, 12, p
17	$\{q, \neg s\}$	resolve 5, 6, r
18	$\{p, r\}$	resolve 5, 7, s
19	$\{\neg s\}$	resolve 5, 8, r
20	$\{s, \neg s\}$	resolve 5, 10, r
21	$\{r, \neg r\}$	resolve 5, 10, s
22	$\{\neg q, \neg s\}$	resolve 5, 16, r
23	$\{q, \neg q\}$	resolve 6, 11, r
24	$\{p, \neg p\}$	resolve 7, 9, s
25	$\{p\}$	resolve 7, 19, s
26	$\{\neg q\}$	resolve 8, 11, r

no further resolvents

$\Longrightarrow$

satisfiable

Example ❸

$$(p \vee q) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee s) \wedge (p \vee \neg r) \wedge (r \vee \neg s)$$

1	$\{p, q\}$	
2	$\{\neg p, \neg r\}$	
3	$\{\neg q, s\}$	
4	$\{p, \neg r\}$	
5	$\{r, \neg s\}$	
6	$\{q, \neg r\}$	resolve 1, 2, p
7	$\{p, s\}$	resolve 1, 3, q
8	$\{\neg r\}$	resolve 2, 4, p
9	$\{\neg p, \neg s\}$	resolve 2, 5, r
10	$\{\neg r, s\}$	resolve 2, 7, p
11	$\{\neg q, r\}$	resolve 3, 5, s
12	$\{\neg p, \neg q\}$	resolve 3, 9, s
13	$\{p, \neg s\}$	resolve 4, 5, r
14	$\{\neg r, \neg s\}$	resolve 4, 9, p
15	$\{p, \neg q\}$	resolve 4, 11, r
16	$\{\neg q, \neg r\}$	resolve 4, 12, p
17	$\{q, \neg s\}$	resolve 5, 6, r
18	$\{p, r\}$	resolve 5, 7, s
19	$\{\neg s\}$	resolve 5, 8, r
20	$\{s, \neg s\}$	resolve 5, 10, r
21	$\{r, \neg r\}$	resolve 5, 10, s
22	$\{\neg q, \neg s\}$	resolve 5, 16, r
23	$\{q, \neg q\}$	resolve 6, 11, r
24	$\{p, \neg p\}$	resolve 7, 9, s
25	$\{p\}$	resolve 7, 19, s
26	$\{\neg q\}$	resolve 8, 11, r

no further resolvents

$\Longrightarrow$

satisfiable

Example ❹

$$(\neg p \wedge \neg q) \vee (s \wedge u) \vee (r \wedge w) \vee (\neg t \wedge \neg u) \vee (p \wedge r) \vee (q \wedge s)$$
$$\vee (p \wedge t) \vee (q \wedge u) \vee (\neg r \wedge \neg s) \vee (t \wedge v) \vee (\neg v \wedge \neg w)$$

Example ❹

$$\neg\big(\,(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge\, (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)\,\big)$$

Example ④

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$
2 $\{\neg s, \neg u\}$
3 $\{\neg r, \neg w\}$
4 $\{t, u\}$
5 $\{\neg p, \neg r\}$
6 $\{\neg q, \neg s\}$
7 $\{\neg p, \neg t\}$
8 $\{\neg q, \neg u\}$
9 $\{r, s\}$
10 $\{\neg t, \neg v\}$
11 $\{v, w\}$

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$

2 $\{\neg s, \neg u\}$

3 $\{\neg r, \neg w\}$

4 $\{t, u\}$

5 $\{\neg p, \neg r\}$

6 $\{\neg q, \neg s\}$

7 $\{\neg p, \neg t\}$

8 $\{\neg q, \neg u\}$

9 $\{r, s\}$

10 $\{\neg t, \neg v\}$

11 $\{v, w\}$

12 $\{\neg s, t\}$ resolve 2, 4, u

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$

2 $\{\neg s, \neg u\}$

3 $\{\neg r, \neg w\}$

4 $\{t, u\}$

5 $\{\neg p, \neg r\}$

6 $\{\neg q, \neg s\}$

7 $\{\neg p, \neg t\}$

8 $\{\neg q, \neg u\}$

9 $\{r, s\}$

10 $\{\neg t, \neg v\}$

11 $\{v, w\}$

12 $\{\neg s, t\}$

13 $\{\neg p, s\}$ resolve 5, 9, r

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$

2 $\{\neg s, \neg u\}$

3 $\{\neg r, \neg w\}$

4 $\{t, u\}$

5 $\{\neg p, \neg r\}$

6 $\{\neg q, \neg s\}$

7 $\{\neg p, \neg t\}$

8 $\{\neg q, \neg u\}$

9 $\{r, s\}$

10 $\{\neg t, \neg v\}$

11 $\{v, w\}$

12 $\{\neg s, t\}$

13 $\{\neg p, s\}$

14 $\{\neg p, \neg s\}$ resolve 7, 12, t

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$

2 $\{\neg s, \neg u\}$

3 $\{\neg r, \neg w\}$

4 $\{t, u\}$

5 $\{\neg p, \neg r\}$

6 $\{\neg q, \neg s\}$

7 $\{\neg p, \neg t\}$

8 $\{\neg q, \neg u\}$

9 $\{r, s\}$

10 $\{\neg t, \neg v\}$

11 $\{v, w\}$

12 $\{\neg s, t\}$

13 $\{\neg p, s\}$

14 $\{\neg p, \neg s\}$

15 $\{\neg p\}$ resolve 13, 14, s

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$

2 $\{\neg s, \neg u\}$

3 $\{\neg r, \neg w\}$

4 $\{t, u\}$

5 $\{\neg p, \neg r\}$

6 $\{\neg q, \neg s\}$

7 $\{\neg p, \neg t\}$

8 $\{\neg q, \neg u\}$

9 $\{r, s\}$

10 $\{\neg t, \neg v\}$

11 $\{v, w\}$

12 $\{\neg s, t\}$

13 $\{\neg p, s\}$

14 $\{\neg p, \neg s\}$

15 $\{\neg p\}$

16 $\{q\}$ resolve 1, 15, p

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$
2 $\{\neg s, \neg u\}$
3 $\{\neg r, \neg w\}$
4 $\{t, u\}$
5 $\{\neg p, \neg r\}$
6 $\{\neg q, \neg s\}$
7 $\{\neg p, \neg t\}$
8 $\{\neg q, \neg u\}$
9 $\{r, s\}$
10 $\{\neg t, \neg v\}$
11 $\{v, w\}$
12 $\{\neg s, t\}$
13 $\{\neg p, s\}$
14 $\{\neg p, \neg s\}$
15 $\{\neg p\}$
16 $\{q\}$
17 $\{\neg s\}$ resolve 6, 16, q

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$
2 $\{\neg s, \neg u\}$
3 $\{\neg r, \neg w\}$
4 $\{t, u\}$
5 $\{\neg p, \neg r\}$
6 $\{\neg q, \neg s\}$
7 $\{\neg p, \neg t\}$
8 $\{\neg q, \neg u\}$
9 $\{r, s\}$
10 $\{\neg t, \neg v\}$
11 $\{v, w\}$
12 $\{\neg s, t\}$
13 $\{\neg p, s\}$
14 $\{\neg p, \neg s\}$
15 $\{\neg p\}$
16 $\{q\}$
17 $\{\neg s\}$
18 $\{r\}$ resolve 9, 17, s

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s) \wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$
2 $\{\neg s, \neg u\}$
3 $\{\neg r, \neg w\}$
4 $\{t, u\}$
5 $\{\neg p, \neg r\}$
6 $\{\neg q, \neg s\}$
7 $\{\neg p, \neg t\}$
8 $\{\neg q, \neg u\}$
9 $\{r, s\}$
10 $\{\neg t, \neg v\}$
11 $\{v, w\}$
12 $\{\neg s, t\}$
13 $\{\neg p, s\}$
14 $\{\neg p, \neg s\}$
15 $\{\neg p\}$
16 $\{q\}$
17 $\{\neg s\}$
18 $\{r\}$
19 $\{\neg w\}$ resolve 3, 18, r

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$
2 $\{\neg s, \neg u\}$
3 $\{\neg r, \neg w\}$
4 $\{t, u\}$
5 $\{\neg p, \neg r\}$
6 $\{\neg q, \neg s\}$
7 $\{\neg p, \neg t\}$
8 $\{\neg q, \neg u\}$
9 $\{r, s\}$
10 $\{\neg t, \neg v\}$
11 $\{v, w\}$
12 $\{\neg s, t\}$
13 $\{\neg p, s\}$
14 $\{\neg p, \neg s\}$
15 $\{\neg p\}$
16 $\{q\}$
17 $\{\neg s\}$
18 $\{r\}$
19 $\{\neg w\}$
20 $\{v\}$ resolve 11, 19, w

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$	10 $\{\neg t, \neg v\}$	19 $\{\neg w\}$
2 $\{\neg s, \neg u\}$	11 $\{v, w\}$	20 $\{v\}$
3 $\{\neg r, \neg w\}$	12 $\{\neg s, t\}$	21 $\{\neg t\}$ resolve 10, 20, v
4 $\{t, u\}$	13 $\{\neg p, s\}$	
5 $\{\neg p, \neg r\}$	14 $\{\neg p, \neg s\}$	
6 $\{\neg q, \neg s\}$	15 $\{\neg p\}$	
7 $\{\neg p, \neg t\}$	16 $\{q\}$	
8 $\{\neg q, \neg u\}$	17 $\{\neg s\}$	
9 $\{r, s\}$	18 $\{r\}$	

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$
2 $\{\neg s, \neg u\}$
3 $\{\neg r, \neg w\}$
4 $\{t, u\}$
5 $\{\neg p, \neg r\}$
6 $\{\neg q, \neg s\}$
7 $\{\neg p, \neg t\}$
8 $\{\neg q, \neg u\}$
9 $\{r, s\}$
10 $\{\neg t, \neg v\}$
11 $\{v, w\}$
12 $\{\neg s, t\}$
13 $\{\neg p, s\}$
14 $\{\neg p, \neg s\}$
15 $\{\neg p\}$
16 $\{q\}$
17 $\{\neg s\}$
18 $\{r\}$
19 $\{\neg w\}$
20 $\{v\}$
21 $\{\neg t\}$
22 $\{u\}$ resolve 4, 21, t

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1	$\{p, q\}$	10	$\{\neg t, \neg v\}$	19	$\{\neg w\}$
2	$\{\neg s, \neg u\}$	11	$\{v, w\}$	20	$\{v\}$
3	$\{\neg r, \neg w\}$	12	$\{\neg s, t\}$	21	$\{\neg t\}$
4	$\{t, u\}$	13	$\{\neg p, s\}$	22	$\{u\}$
5	$\{\neg p, \neg r\}$	14	$\{\neg p, \neg s\}$	23	$\{\neg q\}$ resolve 8, 22, u
6	$\{\neg q, \neg s\}$	15	$\{\neg p\}$		
7	$\{\neg p, \neg t\}$	16	$\{q\}$		
8	$\{\neg q, \neg u\}$	17	$\{\neg s\}$		
9	$\{r, s\}$	18	$\{r\}$		

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$
2 $\{\neg s, \neg u\}$
3 $\{\neg r, \neg w\}$
4 $\{t, u\}$
5 $\{\neg p, \neg r\}$
6 $\{\neg q, \neg s\}$
7 $\{\neg p, \neg t\}$
8 $\{\neg q, \neg u\}$
9 $\{r, s\}$
10 $\{\neg t, \neg v\}$
11 $\{v, w\}$
12 $\{\neg s, t\}$
13 $\{\neg p, s\}$
14 $\{\neg p, \neg s\}$
15 $\{\neg p\}$
16 $\{q\}$
17 $\{\neg s\}$
18 $\{r\}$
19 $\{\neg w\}$
20 $\{v\}$
21 $\{\neg t\}$
22 $\{u\}$
23 $\{\neg q\}$
24 $\Box$ resolve 16, 23, q

Example ❹

$$(p \vee q) \wedge (\neg s \vee \neg u) \wedge (\neg r \vee \neg w) \wedge (t \vee u) \wedge (\neg p \vee \neg r) \wedge (\neg q \vee \neg s)$$
$$\wedge (\neg p \vee \neg t) \wedge (\neg q \vee \neg u) \wedge (r \vee s) \wedge (\neg t \vee \neg v) \wedge (v \vee w)$$

1 $\{p, q\}$
2 $\{\neg s, \neg u\}$
3 $\{\neg r, \neg w\}$
4 $\{t, u\}$
5 $\{\neg p, \neg r\}$
6 $\{\neg q, \neg s\}$
7 $\{\neg p, \neg t\}$
8 $\{\neg q, \neg u\}$
9 $\{r, s\}$
10 $\{\neg t, \neg v\}$
11 $\{v, w\}$
12 $\{\neg s, t\}$
13 $\{\neg p, s\}$
14 $\{\neg p, \neg s\}$
15 $\{\neg p\}$
16 $\{q\}$
17 $\{\neg s\}$
18 $\{r\}$
19 $\{\neg w\}$
20 $\{v\}$
21 $\{\neg t\}$
22 $\{u\}$
23 $\{\neg q\}$
24 $\square$

unsatisfiable

Example ④

$$(\neg p \wedge \neg q) \vee (s \wedge u) \vee (r \wedge w) \vee (\neg t \wedge \neg u) \vee (p \wedge r) \vee (q \wedge s)$$
$$\vee (p \wedge t) \vee (q \wedge u) \vee (\neg r \wedge \neg s) \vee (t \wedge v) \vee (\neg v \wedge \neg w)$$

1	$\{p, q\}$	10	$\{\neg t, \neg v\}$	19	$\{\neg w\}$
2	$\{\neg s, \neg u\}$	11	$\{v, w\}$	20	$\{v\}$
3	$\{\neg r, \neg w\}$	12	$\{\neg s, t\}$	21	$\{\neg t\}$
4	$\{t, u\}$	13	$\{\neg p, s\}$	22	$\{u\}$
5	$\{\neg p, \neg r\}$	14	$\{\neg p, \neg s\}$	23	$\{\neg q\}$
6	$\{\neg q, \neg s\}$	15	$\{\neg p\}$	24	$\square$
7	$\{\neg p, \neg t\}$	16	$\{q\}$		valid
8	$\{\neg q, \neg u\}$	17	$\{\neg s\}$		
9	$\{r, s\}$	18	$\{r\}$		

Outline

1. **Summary of Previous Lecture**
2. **Completeness**
3. **Resolution**
4. **Intermezzo**
5. **Binary Decision Diagrams**
6. **Further Reading**

Question

Which clauses can be obtained by resolving two clauses from the following clausal form ?

$$\{\{p, q, \neg q\}, \{r, \neg q\}, \{p, \neg r\}, \{q, \neg s\}, \{\neg r, s\}\}$$

A $\{p, s\}$

B $\{p, \neg q\}$

C $\{\neg r, q, \neg q\}$

D $\{r, \neg r\}$

E $\{p, q, \neg s\}$

F $\{p\}$

Outline

Definitions

- boolean function f of n arguments is mapping from $\{0,1\}^n$ to $\{0,1\}$

Definitions

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- four basic functions

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0	1	0
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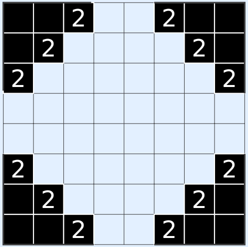
Remarks

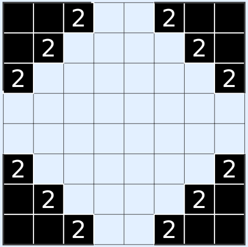
- every boolean function can be expressed in terms of basic functions
- propositional formulas and truth tables are different representations of boolean functions

Exclusive Or or Or



Shakashaka

Exclusive Or or Or



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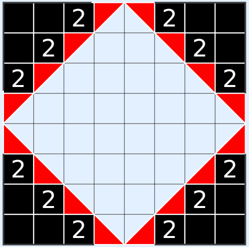

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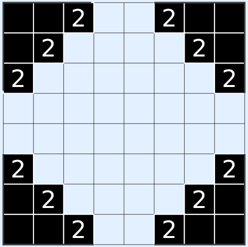

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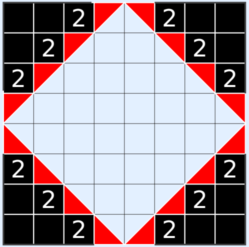


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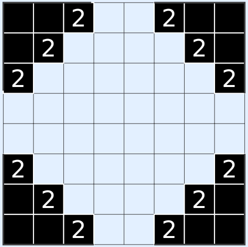

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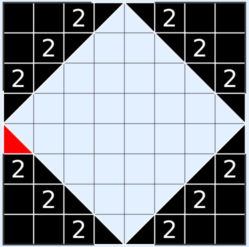


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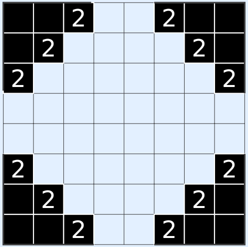

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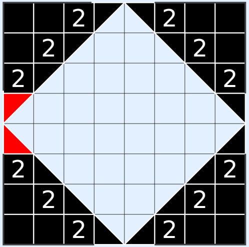


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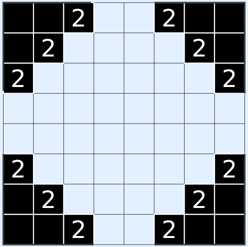

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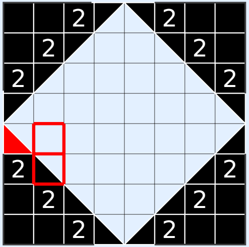


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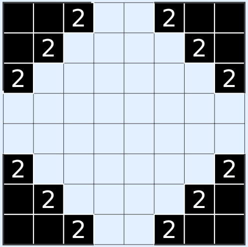

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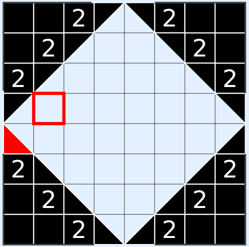


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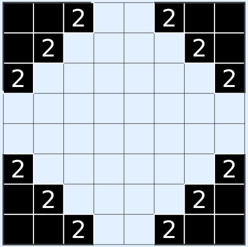

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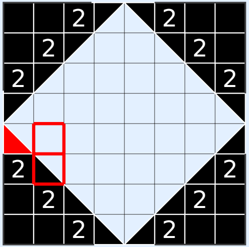


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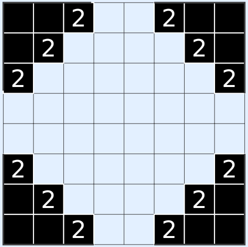

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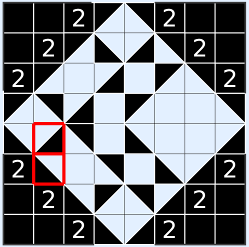


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? = reduced ordered binary decision diagrams

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- majority function

$$f(x,y,z) = \begin{cases} 1 & \text{if } x+y+z > 1 \\ 0 & \text{otherwise} \end{cases}$$

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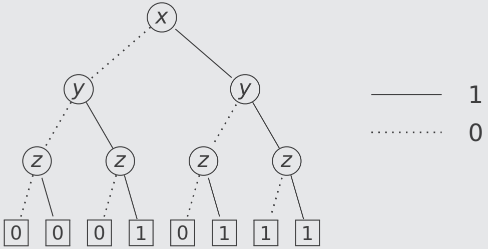

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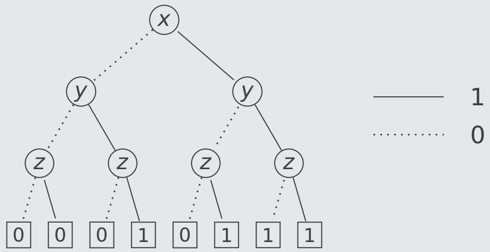

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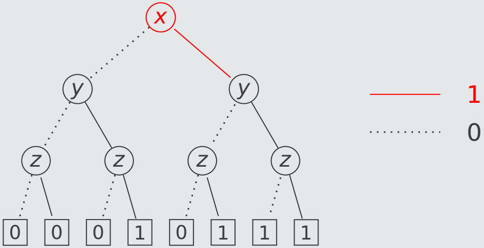

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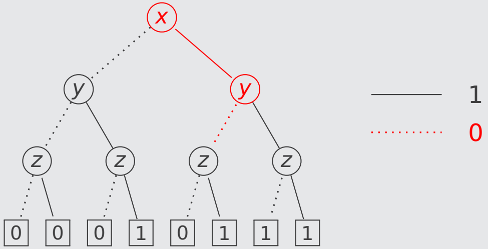

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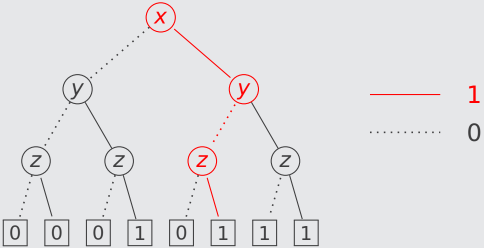

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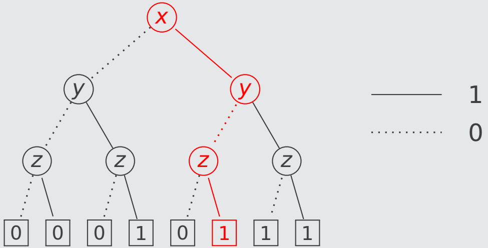

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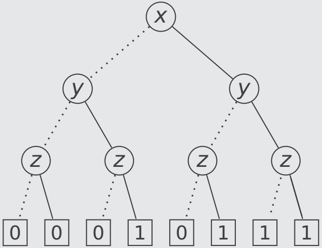

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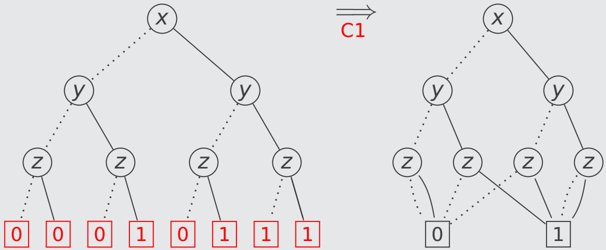
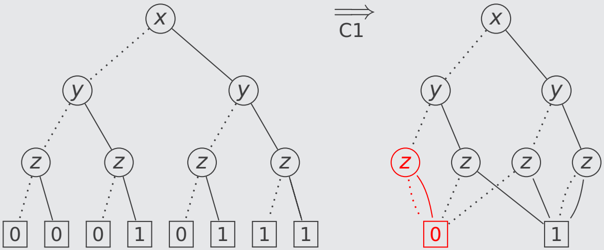


Optimisation Rules

C1 remove duplicate terminals

Example (Binary Decision Diagram)



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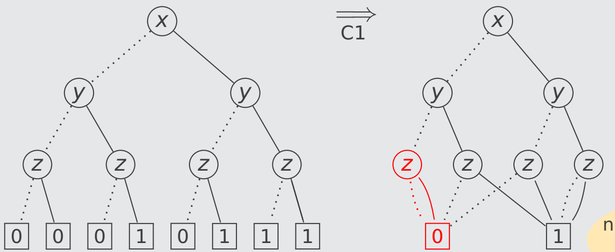

non-terminal with both outgoing edges pointing to same node

Optimisation Rules

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- C2 remove redundant tests

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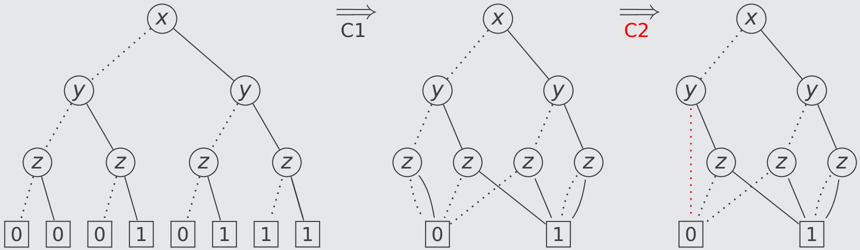

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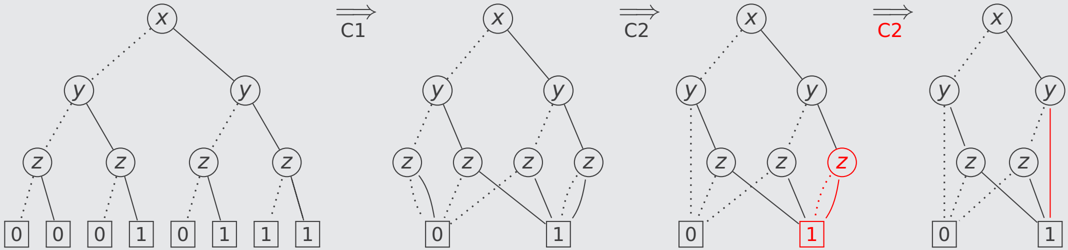

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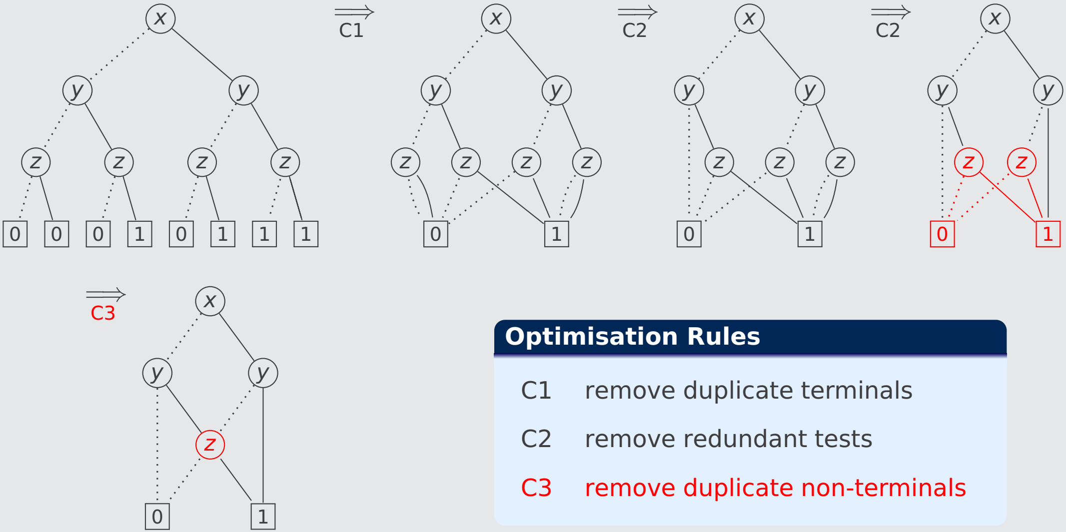

Optimisation Rules

C1	remove duplicate terminals
C2	remove redundant tests
C3	remove duplicate non-terminals

Remark

binary decision diagram (BDD) is directed acyclic graph (dag)

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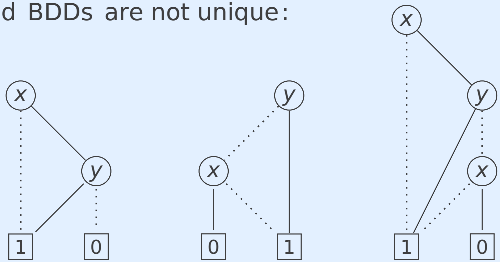
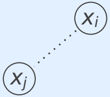
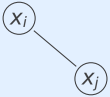
reduced BDDs are not unique:



represent boolean function $\overline{x} + y$

Definition

BDD B is ordered if there exists order $[x_1, \ldots, x_n]$ of variables in B such that

$i < j$ for all edges  and  in B

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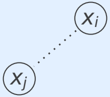
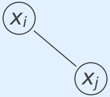
$i < j$ for all edges  and  in B

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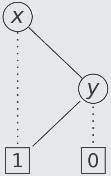
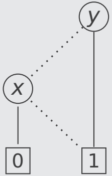


OBDD

$[x, y]$



OBDD

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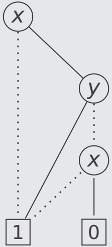

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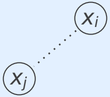
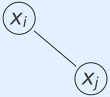
$i < j$ for all edges  and  in B

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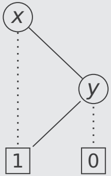

OBDD

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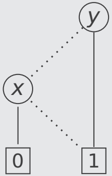

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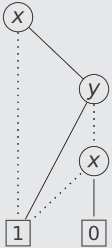

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four variable orders

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- o_2 and o_4

Definition

orders o_1 and o_2 are compatible if o_1 and o_2 are subsequences of some order o

Example

four variable orders

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Lemma

reductions C1, C2, C3 preserve order

Theorem

reduced OBDD representation of boolean function for given order is unique

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Corollary

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- validity

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- validity
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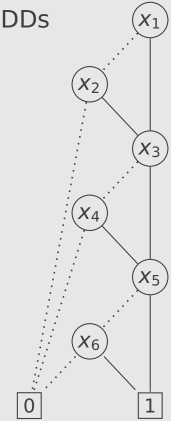
is trivial for reduced OBDDs with compatible variable orders

Example

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 + x_2) \cdot (x_3 + x_4) \cdot (x_5 + x_6)$$

different reduced OBDDs



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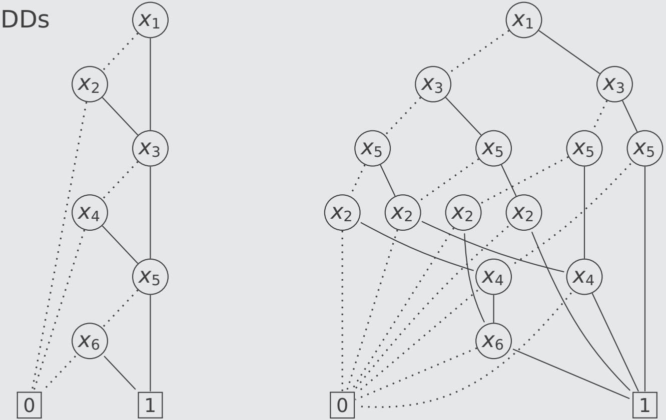

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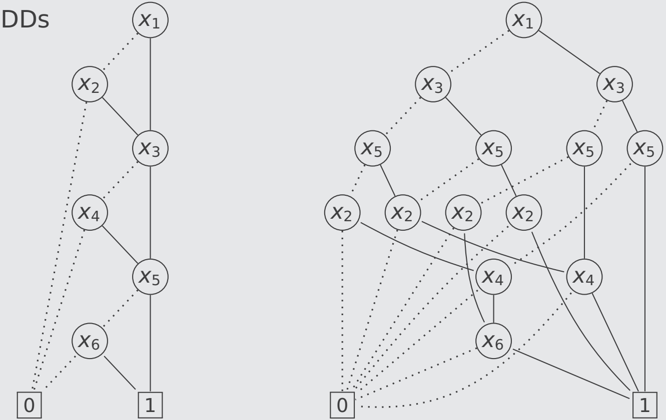

$[x_1, x_2, x_3, x_4, x_5, x_6]$ $[x_1, x_3, x_5, x_2, x_4, x_6]$

Outline

Huth and Ryan

- Section 1.4.4
- Section 6.1

Huth and Ryan

- Section 1.4.4
- Section 6.1

Resolution

- Wikipedia [accessed December 7, 2022]

Important Concepts

- binary decision diagram
- binary decision tree
- boolean function
- complementary literals
- completeness
- clashing
- clausal form
- clause
- compatible variable order
- empty clause
- exclusive or
- ordered BDD
- parent clauses
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- refutation
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homework for April 11