SS 2024 lecture 6

Logic

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with session ID **0992 9580** for anonymous questions

Outline

BDD Algorithms

- reduce
  - input: • OBDD
  - output: • equivalent reduced OBDD with compatible variable ordering
- restrict
  - input: • OBDD B_f, variable x, $i \in \{0,1\}$
  - output: • reduced OBDD of $f[i/x]$ with compatible variable ordering
- apply
  - input: • binary operation $\star$ on boolean functions
    • OBDDs B_f and B_g with compatible variable orderings
  - output: • reduced OBDD of $f \star g$ with compatible variable ordering

Theorem (Shannon expansion)

$f = \overline{x} \cdot f[0/x] + x \cdot f[1/x]$ for every boolean function f and variable x

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Theorem (Shannon expansion)

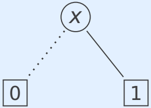
$f = \overline{x} \cdot f[0/x] + x \cdot f[1/x] = \overline{x} \cdot f[0/x] \oplus x \cdot f[1/x]$ for every boolean function f and variable x

Definition

quantification of boolean function f over variable x:

$$\exists x.f = f[0/x] + f[1/x] \qquad\qquad \forall x.f = f[0/x] \cdot f[1/x]$$

BDD operations

function f	OBDD B_f	function f	OBDD B_f	function f	OBDD B_f
0	$\boxed{0}$	$g + h$	$\text{apply}(+, B_g, B_h)$	$g[0/x]$	$\text{restrict}(0, x, B_g)$
1	$\boxed{1}$	$g \oplus h$	$\text{apply}(\oplus, B_g, B_h)$	$g[1/x]$	$\text{restrict}(1, x, B_g)$
x		$g \cdot h$	$\text{apply}(\cdot, B_g, B_h)$	$\exists x.g$	$\text{apply}(+, B_{g[0/x]}, B_{g[1/x]})$
		$\overline{g}$	$\text{apply}(\oplus, B_g, B_1)$	$\forall x.g$	$\text{apply}(\cdot, B_{g[0/x]}, B_{g[1/x]})$

Remark

(reduced ordered) BDDs are not always efficient representation

hidden weighted bit function multiplication

Definitions

- terms in predicate logic are built from function symbols and variables according to BNF grammar $t ::= x \mid c \mid f(t, \dots, t)$
- formulas in predicate logic are built according to BNF grammar

$$\varphi ::= P \mid P(t, \dots, t) \mid t = t \mid \bot \mid \top \mid (\neg \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\forall x\, \varphi) \mid (\exists x\, \varphi)$$

- occurrence of variable x in formula φ is free in φ if it is leaf node in parse tree of φ such that there is no node $\forall x$ or $\exists x$ on path to root node; all other occurrences of x are bound
- $\varphi[t/x]$ is result of replacing all free occurrences of x in φ by t
- t is free for x in φ if variables in t do not become bound in $\varphi[t/x]$
- sentence is formula without free variables

Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, soundness and completeness, syntax, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching-time temporal logic, CTL^*, fairness, linear-time temporal logic, model checking algorithms, symbolic model checking

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④ $=^{\mathcal{M}}$ is identity relation on A

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Examples

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- $\mathcal{P}$ A, B: arity 2 P, S, L: arity 1 $\mathcal{F}$ m: arity 0

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1. model $\mathcal{M}_1$
   - universe A_1: set of computer science students and professors of University of Innsbruck together with all lectures offered in SS 2024 in bachelor program computer science
   - $A^{\mathcal{M}_1} = \{(x,y) \mid x \text{ admires } y\}$ $\quad P^{\mathcal{M}_1} = \{x \mid x \text{ is professor}\}$ $\quad L^{\mathcal{M}_1} = \{x \mid x \text{ is lecture}\}$
     $B^{\mathcal{M}_1} = \{(x,y) \mid x \text{ attended } y\}$ $\quad S^{\mathcal{M}_1} = \{x \mid x \text{ is student}\}$ $\quad m^{\mathcal{M}_1} = \text{Aki Suzuki}$

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2. model $\mathcal{M}_2$
   - universe A_2: set of natural numbers
   - $A^{\mathcal{M}_2} = \{(x,y) \mid x > y\}$ $P^{\mathcal{M}_2} = \{x \mid x \text{ is prime number}\}$ $L^{\mathcal{M}_2} = \{2, 7, 111\}$
     $B^{\mathcal{M}_2} = \{(x,y) \mid x + y = 5\}$ $S^{\mathcal{M}_2} = \{x^2 \mid x > 1\}$ $m^{\mathcal{M}_2} = 13$

Examples

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   - $A^{\mathcal{M}_1} = \{(x,y) \mid x \text{ admires } y\}$ $P^{\mathcal{M}_1} = \{x \mid x \text{ is professor}\}$ $L^{\mathcal{M}_1} = \{x \mid x \text{ is lecture}\}$
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Definitions

- environment (look-up table) for model $\mathcal{M} = (A, \{f^{\mathcal{M}}\}_{f \in \mathcal{F}}, \{P^{\mathcal{M}}\}_{P \in \mathcal{P}})$ is mapping l from variables to elements of A

Definitions

- environment (look-up table) for model $\mathcal{M} = (A, \{f^{\mathcal{M}}\}_{f \in \mathcal{F}}, \{P^{\mathcal{M}}\}_{P \in \mathcal{P}})$ is mapping I from variables to elements of A
- value $t^{\mathcal{M},I}$ of term t in model $\mathcal{M}$ relative to environment I is defined inductively:

$$t^{\mathcal{M},I} = \begin{cases} I(t) & \text{if } t \text{ is variable} \\ \end{cases}$$

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- value $t^{\mathcal{M},l}$ of term t in model $\mathcal{M}$ relative to environment l is defined inductively:

$$t^{\mathcal{M},l} = \begin{cases} l(t) & \text{if } t \text{ is variable} \\ f^{\mathcal{M}}(t_1^{\mathcal{M},l}, \dots, t_n^{\mathcal{M},l}) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

- given environment l, variable x, and element $a \in A$, environment $l[x \mapsto a]$ is defined as

$$l[x \mapsto a](y) = \begin{cases} a & \text{if } y = x \\ l(y) & \text{if } y \neq x \end{cases}$$

Example

function symbols $\mathcal{F}$

- f: arity 2 $\quad g, h$: arity 1 $\quad a$: arity 0

model $\mathcal{M}$

- universe A: set of natural numbers
- $f^{\mathcal{M}}(x,y) = x \times y \qquad g^{\mathcal{M}}(x) = x + 1 \qquad h^{\mathcal{M}}(x) = x^2 \qquad a^{\mathcal{M}} = 2$

environment I

- $I(x) = 3 \qquad I(y) = 5 \qquad \ldots$

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$$f(x,g(y))^{\mathcal{M},l} = 18$$

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$$f(x,g(y))^{\mathcal{M},l} = 18 \qquad f(x,g(f(x,h(x))))^{\mathcal{M},l} = 84$$

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$$f(x,g(y))^{\mathcal{M},I} = 18 \qquad f(x,g(f(x,h(x))))^{\mathcal{M},I} = 84 \qquad f(h(a),g(f(a,h(h(a)))))^{\mathcal{M},I} = ?$$

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$$f(x,g(y))^{\mathcal{M},I} = 18 \qquad f(x,g(f(x,h(x))))^{\mathcal{M},I} = 84 \qquad f(h(a),g(f(a,h(h(a)))))^{\mathcal{M},I} = 132$$

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$$\mathcal{M} \vDash_I \varphi \iff \begin{cases} (t_1^{\mathcal{M},I}, \dots, t_n^{\mathcal{M},I}) \in P^{\mathcal{M}} & \text{if } \varphi = P(t_1, \dots, t_n) \\ t_1^{\mathcal{M},I} = t_2^{\mathcal{M},I} & \text{if } \varphi = (t_1 = t_2) \\ \mathcal{M} \nvDash_I \psi & \text{if } \varphi = \neg\psi \\ \mathcal{M} \vDash_I \psi_1 \text{ and } \mathcal{M} \vDash_I \psi_2 & \text{if } \varphi = \psi_1 \wedge \psi_2 \\ \mathcal{M} \vDash_I \psi_1 \text{ or } \mathcal{M} \vDash_I \psi_2 & \text{if } \varphi = \psi_1 \vee \psi_2 \\ \mathcal{M} \nvDash_I \psi_1 \text{ or } \mathcal{M} \vDash_I \psi_2 & \text{if } \varphi = \psi_1 \rightarrow \psi_2 \\ \mathcal{M} \vDash_{I[x \mapsto a]} \psi \text{ for all } a \in A & \text{if } \varphi = \forall x\, \psi \end{cases}$$

Notation

$\mathcal{M} \nvDash_I \psi$ denotes "not $\mathcal{M} \vDash_I \psi$"

Definition

satisfaction relation $\mathcal{M} \vDash_I \varphi$ (model $\mathcal{M}$, enviroment I, formula φ) is defined inductively:

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$$\mathcal{M} \nvDash_I \bot$$

$$\mathcal{M} \vDash_I \varphi \iff \begin{cases} (t_1^{\mathcal{M},I}, \ldots, t_n^{\mathcal{M},I}) \in P^{\mathcal{M}} & \text{if } \varphi = P(t_1, \ldots, t_n) \\ t_1^{\mathcal{M},I} = t_2^{\mathcal{M},I} & \text{if } \varphi = (t_1 = t_2) \\ \mathcal{M} \nvDash_I \psi & \text{if } \varphi = \neg\psi \\ \mathcal{M} \vDash_I \psi_1 \text{ and } \mathcal{M} \vDash_I \psi_2 & \text{if } \varphi = \psi_1 \wedge \psi_2 \\ \mathcal{M} \vDash_I \psi_1 \text{ or } \mathcal{M} \vDash_I \psi_2 & \text{if } \varphi = \psi_1 \vee \psi_2 \\ \mathcal{M} \nvDash_I \psi_1 \text{ or } \mathcal{M} \vDash_I \psi_2 & \text{if } \varphi = \psi_1 \rightarrow \psi_2 \\ \mathcal{M} \vDash_{I[x \mapsto a]} \psi \text{ for all } a \in A & \text{if } \varphi = \forall x\, \psi \\ \mathcal{M} \vDash_{I[x \mapsto a]} \psi \text{ for some } a \in A & \text{if } \varphi = \exists x\, \psi \end{cases}$$

Notation

$\mathcal{M} \nvDash_I \psi$ denotes "not $\mathcal{M} \vDash_I \psi$"

Definition

sentence is formula without free variables

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Lemma

if φ is sentence then

$$\mathcal{M} \vDash_{I} \varphi \iff \mathcal{M} \vDash_{I'} \varphi$$

for all environments I and I'

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$$\mathcal{M} \vDash_I \varphi \quad\Longleftrightarrow\quad \mathcal{M} \vDash_{I'} \varphi$$

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truth value of sentence does not depend on environment

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sentence is formula without free variables

Lemma

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for all environments I and I'

truth value of sentence does not depend on environment

Notation

$\mathcal{M} \vDash \varphi$ instead of $\mathcal{M} \vDash_I \varphi$ for sentences φ

Example

- function and predicate symbols $\mathcal{P}$ R: arity 2 $\mathcal{F}$ f: arity 1 a: arity 0

Example

- function and predicate symbols $\mathcal{P}$ R: arity 2 $\mathcal{F}$ f: arity 1 a: arity 0
- model $\mathcal{M}_1$: universe $A_1 = \mathbb{N}$ $R^{\mathcal{M}_1} = \{(x,y) \mid x < y\}$ $f^{\mathcal{M}_1}(x) = 2x$ $a^{\mathcal{M}_1} = 0$

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$$\varphi_1 = \exists\, x\, R(a,x)$$

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$$\varphi_1 = \exists\, x\, R(a,x) \qquad \mathcal{M}_1 \vDash \varphi_1 \quad \mathcal{M}_2 \vDash \varphi_1 \quad \mathcal{M}_3 \vDash \varphi_1$$

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$$\varphi_3 = \forall\, x\, \forall\, y\, (R(x,y) \rightarrow \exists\, z\, (R(x,z) \wedge R(z,y))) \qquad \mathcal{M}_1 \nvDash \varphi_3 \quad \mathcal{M}_2 \vDash \varphi_3 \quad \mathcal{M}_3 \nvDash \varphi_3$$

Example

some professor admires Mary

Example

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$$\varphi = \exists\, x\, (P(x) \wedge A(x, m))$$

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- model $\mathcal{M}$: universe is set of persons living in Innsbruck

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  $P^{\mathcal{M}} = \varnothing \qquad A^{\mathcal{M}} = \varnothing \qquad m^{\mathcal{M}} = \text{Diana}$

Example

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- $\mathcal{M} \nvDash \varphi$

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- $\mathcal{M} \nvDash \varphi$
- $\mathcal{M} \vDash \psi$

Definitions

formula ψ

- ψ is satisfiable if $\mathcal{M} \vDash_I \psi$ for some model $\mathcal{M}$ and environment I

Definitions

formula ψ, (possibly infinite) set of formulas Γ

- ψ is satisfiable if $\mathcal{M} \vDash_I \psi$ for some model $\mathcal{M}$ and environment I
- Γ is satisfiable (consistent) if $\mathcal{M} \vDash_I \varphi$ for all $\varphi \in \Gamma$, for some model $\mathcal{M}$ and environment I

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Example

$\Gamma = \{\varphi_1, \varphi_2, \varphi_3\}$ with
$$\begin{aligned}
\varphi_1 &= \exists x\, R(a,x) \\
\varphi_2 &= \forall x\, (R(x,f(x)) \vee x = a) \\
\varphi_3 &= \forall x\, \forall y\, (R(x,y) \to \exists z\, (R(x,z) \wedge R(z,y)))
\end{aligned}$$

is satisfiable

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$$\Gamma = \{\varphi_1, \varphi_2, \varphi_3\} \text{ with } \begin{aligned} \varphi_1 &= \exists\, x\, R(a,x) \\ \varphi_2 &= \forall\, x\, (R(x,f(x)) \vee x = a) \\ \varphi_3 &= \forall\, x\, \forall\, y\, (R(x,y) \to \exists\, z\, (R(x,z) \wedge R(z,y))) \end{aligned}$$

is satisfiable in model $\mathcal{M}$:

- universe A: set of natural numbers
- $R^{\mathcal{M}} = \{(x,y) \mid x \leqslant y\}$ $\quad f^{\mathcal{M}}(x) = x$ $\quad a^{\mathcal{M}} = 0$

Definitions

formula ψ, (possibly infinite) set of formulas Γ

- $\Gamma \vDash \psi$ (semantic entailment) if $\mathcal{M} \vDash_I \psi$ whenever $\mathcal{M} \vDash_I \varphi$ for all $\varphi \in \Gamma$, for all (appropriate) models $\mathcal{M}$ and environments I

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Example

- $\Gamma \vDash \neg R(a, a) \rightarrow \exists x \neg (x = a)$ for $\Gamma = \{\varphi_1, \varphi_2, \varphi_3\}$ with

$$\begin{aligned} \varphi_1 &= \exists x\, R(a, x) \\ \varphi_2 &= \forall x\, (R(x, f(x)) \vee x = a) \\ \varphi_3 &= \forall x\, \forall y\, (R(x, y) \rightarrow \exists z\, (R(x, z) \wedge R(z, y))) \end{aligned}$$

Definitions

formula ψ, (possibly infinite) set of formulas Γ

- $\Gamma \vDash \psi$ (semantic entailment) if $\mathcal{M} \vDash_l \psi$ whenever $\mathcal{M} \vDash_l \varphi$ for all $\varphi \in \Gamma$, for all (appropriate) models $\mathcal{M}$ and environments l
- ψ is valid if $\mathcal{M} \vDash_l \psi$ for all (appropriate) models $\mathcal{M}$ and environments l

Example

- $\Gamma \vDash \neg R(a, a) \rightarrow \exists\, x\, \neg(x = a)$ for $\Gamma = \{\varphi_1, \varphi_2, \varphi_3\}$ with

$$
\begin{aligned}
\varphi_1 &= \exists\, x\, R(a, x) \\
\varphi_2 &= \forall\, x\, (R(x, f(x)) \vee x = a) \\
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\end{aligned}
$$

Definitions

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\varphi_3 &= \forall x\, \forall y\, (R(x,y) \rightarrow \exists z\, (R(x,z) \wedge R(z,y)))
\end{aligned}$$

- $\forall x\, \forall y\, (x = y \rightarrow f(x) = f(y))$ is valid

Outline

Question

Which of the following statements are true ?

A The semantic entailment $\forall x\, \varphi \vDash \exists x\, \varphi$ holds for all formulas φ.

B The formulas $\exists x\, \forall y\, Q(x, y)$ and $\forall y\, Q(a, y)$ are equisatisfiable.

C The set $\{\forall x\, (P(x) \to \bot), \exists y\, P(y)\}$ is consistent.

D The semantic entailment $x = y \vDash f(x) = f(y)$ holds.

Outline

1. **Summary of Previous Lecture**
2. **Semantics of Predicate Logic**
3. **Intermezzo**
4. **Natural Deduction for Predicate Logic**

   Equality Universal Quantification Existential Quantification

5. **Soundness and Completeness**
6. **Further Reading**

Proof Rules of Natural Deduction ❶

	introduction	elimination
$\wedge$	$\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\varphi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\varphi}{\varphi \vee \psi}\ \vee\mathrm{i}_1 \qquad \dfrac{\psi}{\varphi \vee \psi}\ \vee\mathrm{i}_2$	$\dfrac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \psi\end{array}}}{\varphi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\varphi \rightarrow \psi \quad \varphi}{\psi}\ \rightarrow\mathrm{e}$

Proof Rules of Natural Deduction ❷

	introduction	elimination
$\bot$		$\dfrac{\bot}{\varphi}\ \bot\mathsf{e}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \bot\end{array}}}{\neg\varphi}\ \neg\mathsf{i}$	$\dfrac{\varphi \quad \neg\varphi}{\bot}\ \neg\mathsf{e}$
$\top$	$\dfrac{}{\top}\ \top\mathsf{i}$	
$\neg\neg$		$\dfrac{\neg\neg\varphi}{\varphi}\ \neg\neg\mathsf{e}$

derived proof rules

$$\frac{\varphi \to \psi \quad \neg\psi}{\neg\varphi}\ \mathsf{MT} \qquad \frac{\boxed{\begin{array}{c}\neg\varphi\\ \vdots\\ \bot\end{array}}}{\varphi}\ \mathsf{PBC} \qquad \frac{\varphi}{\neg\neg\varphi}\ \neg\neg\mathsf{i} \qquad \frac{}{\varphi \vee \neg\varphi}\ \mathsf{LEM}$$

Outline

Definitions

- equality introduction

$$\frac{}{t = t}\ {=}\mathrm{i}$$

Definitions

- equality introduction

$$\frac{}{t = t}\ {=}\mathrm{i}$$

- equality elimination "replace equals by equals"

$$\frac{t_1 = t_2 \qquad \varphi[t_1/x]}{\varphi[t_2/x]}\ {=}\mathrm{e}$$

provided t_1 and t_2 are free for x in φ

Examples

1. $s = t \vdash t = s$ is valid:

1	$s = t$	premise
2	$s = s$	$=$i
3	$t = s$	$=$e 1, 2

Examples

1. $s = t \vdash t = s$ is valid:

1	$s = t$	premise	
2	$s = s$	$=$i	
3	$t = s$	$=$e $1, 2$	with $\varphi = (x = s)$, $t_1 = s$, $t_2 = t$

Examples

1. $s = t \vdash t = s$ is valid:

1	$s = t$	premise	
2	$s = s$	$=$i	
3	$t = s$	$=$e 1, 2	with $\varphi = (x = s),\ t_1 = s,\ t_2 = t$

2. $s = t, t = u \vdash s = u$ is valid:

1	$s = t$	premise	
2	$t = u$	premise	
3	$s = u$	$=$e 2, 1	with $\varphi = (s = x),\ t_1 = t,\ t_2 = u$

Outline

Definitions

- $\forall$ elimination

$$\frac{\forall x\, \varphi}{\varphi[t/x]}\ \forall e$$

provided t is free for x in φ

Definitions

- $\forall$ elimination

$$\frac{\forall x\, \varphi}{\varphi[t/x]}\ \forall e$$

provided t is free for x in φ

- $\forall$ introduction

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad\varphi[x_0/x] \end{array}}}{\forall x\, \varphi}\ \forall i$$

where x_0 is fresh variable that is used only inside box

Example

$\forall x\,(P(x) \to Q(x)),\ \forall x\,P(x) \vdash \forall x\,Q(x)$ is valid:

1	$\forall x\,(P(x) \to Q(x))$	premise
2	$\forall x\,P(x)$	premise
	$\forall x\,Q(x)$	

Example

$\forall x\,(P(x) \to Q(x)), \forall x\,P(x) \vdash \forall x\,Q(x)$ is valid:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\forall x\,P(x)$	premise
3	x_0		
		$Q(x_0)$	
		$\forall x\,Q(x)$	$\forall$i

Example

$\forall x\,(P(x) \to Q(x)),\ \forall x\, P(x) \vdash \forall x\, Q(x)$ is valid:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\forall x\, P(x)$	premise
3	x_0	$P(x_0) \to Q(x_0)$	$\forall$e 1
		$Q(x_0)$	
		$\forall x\, Q(x)$	$\forall$i

Example

$\forall x\,(P(x) \to Q(x)),\ \forall x\,P(x) \vdash \forall x\,Q(x)$ is valid:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\forall x\,P(x)$	premise
3	x_0	$P(x_0) \to Q(x_0)$	$\forall$e 1
4		$P(x_0)$	$\forall$e 2
		$Q(x_0)$	
		$\forall x\,Q(x)$	$\forall$i

Example

$\forall x\,(P(x) \to Q(x)),\ \forall x\,P(x) \vdash \forall x\,Q(x)$ is valid:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\forall x\,P(x)$	premise
3	x_0	$P(x_0) \to Q(x_0)$	$\forall$e 1
4		$P(x_0)$	$\forall$e 2
5		$Q(x_0)$	$\to$e 3, 4
		$\forall x\,Q(x)$	$\forall$i

Example

$\forall x\,(P(x) \to Q(x)),\ \forall x\,P(x) \vdash \forall x\,Q(x)$ is valid:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\forall x\,P(x)$	premise
3	x_0	$P(x_0) \to Q(x_0)$	$\forall$e 1
4		$P(x_0)$	$\forall$e 2
5		$Q(x_0)$	$\to$e 3, 4
6		$\forall x\,Q(x)$	$\forall$i 3–5

Example

$P \rightarrow \forall x\, Q(x) \vdash \forall x\, (P \rightarrow Q(x))$ is valid:

1		$P \rightarrow \forall x\, Q(x)$	premise
2	x_0		
3		P	assumption
4		$\forall x\, Q(x)$	$\rightarrow$e 1, 3
5		$Q(x_0)$	$\forall$e 4
6		$P \rightarrow Q(x_0)$	$\rightarrow$i 3–5
7		$\forall x\, (P \rightarrow Q(x))$	$\forall$i 2–6

Outline

Definitions

- ∃ introduction

$$\frac{\varphi[t/x]}{\exists x\, \varphi}\ \exists\text{i}$$

provided t is free for x in φ

Definitions

- $\exists$ introduction

$$\frac{\varphi[t/x]}{\exists x\, \varphi}\ \exists i$$

provided t is free for x in φ

- $\exists$ elimination

$$\frac{\exists x\, \varphi \qquad \boxed{\begin{array}{ll} x_0 & \varphi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists e$$

where x_0 is fresh variable that is used only inside box

Example

$\forall x\,(P(x) \rightarrow Q(x)),\ \exists x\,P(x) \vdash \exists x\,Q(x)$ is valid:

1	$\forall x\,(P(x) \rightarrow Q(x))$	premise
2	$\exists x\,P(x)$	premise
	$\exists x\,Q(x)$	

Example

$\forall x\,(P(x) \to Q(x)),\ \exists x\,P(x) \vdash \exists x\,Q(x)$ is valid:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\exists x\,P(x)$	premise
3	x_0	$P(x_0)$	assumption
		$\exists x\,Q(x)$	
		$\exists x\,Q(x)$	$\exists$ e 2

Example

$\forall x\,(P(x) \to Q(x)),\ \exists x\,P(x) \vdash \exists x\,Q(x)$ is valid:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\exists x\,P(x)$	premise
3	x_0	$P(x_0)$	assumption
4		$P(x_0) \to Q(x_0)$	$\forall$e 1
		$\exists x\,Q(x)$	
		$\exists x\,Q(x)$	$\exists$e 2

Example

$\forall x\,(P(x) \to Q(x)),\ \exists x\,P(x) \vdash \exists x\,Q(x)$ is valid:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\exists x\,P(x)$	premise
3	x_0	$P(x_0)$	assumption
4		$P(x_0) \to Q(x_0)$	$\forall$e 1
5		$Q(x_0)$	$\to$e 4, 3
		$\exists x\,Q(x)$	
		$\exists x\,Q(x)$	$\exists$e 2

Example

$\forall x\,(P(x) \rightarrow Q(x)),\ \exists x\,P(x) \vdash \exists x\,Q(x)$ is valid:

1		$\forall x\,(P(x) \rightarrow Q(x))$	premise
2		$\exists x\,P(x)$	premise
3	x_0	$P(x_0)$	assumption
4		$P(x_0) \rightarrow Q(x_0)$	$\forall$e 1
5		$Q(x_0)$	$\rightarrow$e 4, 3
6		$\exists x\,Q(x)$	$\exists$i 5
		$\exists x\,Q(x)$	$\exists$e 2

Example

$\forall x\,(P(x) \to Q(x)),\ \exists x\, P(x) \vdash \exists x\, Q(x)$ is valid:

1		$\forall x\,(P(x) \to Q(x))$	premise
2		$\exists x\, P(x)$	premise
3	x_0	$P(x_0)$	assumption
4		$P(x_0) \to Q(x_0)$	$\forall$e 1
5		$Q(x_0)$	$\to$e 4, 3
6		$\exists x\, Q(x)$	$\exists$i 5
7		$\exists x\, Q(x)$	$\exists$e 2, 3–6

Lemma

$\forall x\, \varphi \vdash \exists x\, \varphi$ is valid

Lemma

$\forall x\, \varphi \vdash \exists x\, \varphi$ is valid

Proof

1	$\forall x\, \varphi$	premise

Lemma

$\forall x\, \varphi \vdash \exists x\, \varphi$ is valid

Proof

1	$\forall x\, \varphi$	premise
2	$\varphi[x/x]$	$\forall$e 1

Lemma

$\forall x\, \varphi \vdash \exists x\, \varphi$ is valid

Proof

1	$\forall x\, \varphi$	premise
2	$\varphi[x/x]$	$\forall$e 1
3	$\exists x\, \varphi$	$\exists$i 2

Example

$\exists x\, P(x), \forall x \forall y\, (P(x) \rightarrow Q(y)) \vdash \forall y\, Q(y)$ is valid:

1		$\exists x\, P(x)$	premise
2		$\forall x \forall y\, (P(x) \rightarrow Q(y))$	premise
3	y_0		
4	x_0	$P(x_0)$	assumption
5		$\forall y\, (P(x_0) \rightarrow Q(y))$	$\forall$e 2
6		$P(x_0) \rightarrow Q(y_0)$	$\forall$e 5
7		$Q(y_0)$	$\rightarrow$e 6, 4
8		$Q(y_0)$	$\exists$e 1, 4–7
9		$\forall y\, Q(y)$	$\forall$i 3–8

Example

$\exists x\, P(x), \forall x\, \forall y\, (P(x) \to Q(y)) \vdash \forall y\, Q(y)$ is valid:

1		$\exists x\, P(x)$	premise
2		$\forall x\, \forall y\, (P(x) \to Q(y))$	premise
3	z		
4	x_0	$P(x_0)$	assumption
5		$\forall y\, (P(x_0) \to Q(y))$	$\forall$e 2
6		$P(x_0) \to Q(z)$	$\forall$e 5
7		$Q(z)$	$\to$e 6, 4
8		$Q(z)$	$\exists$e 1, 4–7
9		$\forall y\, Q(y)$	$\forall$i 3–8

Example

$\exists x\, P(x), \forall x \forall y\, (P(x) \to Q(y)) \vdash \forall y\, Q(y)$ is valid:

1		$\exists x\, P(x)$	premise
2		$\forall x \forall y\, (P(x) \to Q(y))$	premise
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4	y_0	$P(y_0)$	assumption
5		$\forall y\, (P(y_0) \to Q(y))$	$\forall$e 2
6		$P(y_0) \to Q(z)$	$\forall$e 5
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8		$Q(z)$	$\exists$e 1, 4–7
9		$\forall y\, Q(y)$	$\forall$i 3–8

Lemma

$\neg\forall x\,\varphi \vdash \exists x\,\neg\varphi$ is valid

Lemma

$\neg\forall x\,\varphi \vdash \exists x\,\neg\varphi$ is valid

Proof

1	$\neg\forall x\,\varphi$	premise
2	$\neg\exists x\,\neg\varphi$	assumption
3	x_0	
4	$\neg\varphi[x_0/x]$	assumption
5	$\exists x\,\neg\varphi$	$\exists$i 4
6	$\bot$	$\neg$e 5, 2
7	$\varphi[x_0/x]$	PBC 4–6
8	$\forall x\,\varphi$	$\forall$i 3–7
9	$\bot$	$\neg$e 8, 1
10	$\exists x\,\neg\varphi$	PBC 2–9

Example

$\forall x\, \exists y\, P(x,y),\ \forall x\, \forall y\, (P(x,y) \rightarrow Q(x,y)) \vdash \exists y\, \forall x\, Q(x,y)$ is valid:

1		$\forall x\, \exists y\, P(x,y)$	premise
2		$\forall x\, \forall y\, (P(x,y) \rightarrow Q(x,y))$	premise
3	x_0	$\exists y\, P(x_0,y)$	$\forall$e 1
4		$\forall y\, (P(x_0,y) \rightarrow Q(x_0,y))$	$\forall$e 2
5	y_0	$P(x_0,y_0)$	assumption
6		$P(x_0,y_0) \rightarrow Q(x_0,y_0)$	$\forall$e 4
7		$Q(x_0,y_0)$	$\rightarrow$e 6, 5
8		$Q(x_0,y_0)$	$\exists$e 3, 5–7
9		$\forall x\, Q(x,y_0)$	$\forall$i 3–8
10		$\exists y\, \forall x\, Q(x,y)$	$\exists$i 9

Example

$\forall x \exists y P(x,y), \forall x \forall y (P(x,y) \to Q(x,y)) \vdash \exists y \forall x Q(x,y)$ is not valid:

1		$\forall x \exists y P(x,y)$	premise
2		$\forall x \forall y (P(x,y) \to Q(x,y))$	premise
3	x_0	$\exists y P(x_0,y)$	$\forall$e 1
4		$\forall y (P(x_0,y) \to Q(x_0,y))$	$\forall$e 2
5	y_0	$P(x_0,y_0)$	assumption
6		$P(x_0,y_0) \to Q(x_0,y_0)$	$\forall$e 4
7		$Q(x_0,y_0)$	$\to$e 6, 5
8		$Q(x_0,y_0)$	$\exists$e 3, 5–7
9		$\forall x Q(x,y_0)$	$\forall$i 3–8
10		$\exists y \forall x Q(x,y)$	$\exists$i 9

Example

$\forall x \exists y\, P(x,y),\ \forall x \forall y\, (P(x,y) \rightarrow Q(x,y)) \not\vDash \exists y \forall x\, Q(x,y)$

Example

$\forall x \, \exists y \, P(x,y), \, \forall x \, \forall y \, (P(x,y) \to Q(x,y)) \not\models \exists y \, \forall x \, Q(x,y)$

model $\mathcal{M}$

- universe A: set of natural numbers
- $P^{\mathcal{M}} = Q^{\mathcal{M}} = \{(x,y) \mid x < y\}$

Example

$\forall x \, \exists y \, P(x,y), \; \forall x \, \forall y \, (P(x,y) \to Q(x,y)) \not\models \exists y \, \forall x \, Q(x,y)$

model $\mathcal{M}$

- universe A: set of natural numbers
- $P^{\mathcal{M}} = Q^{\mathcal{M}} = \{(x,y) \mid x < y\}$

$\mathcal{M} \models \forall x \, \exists y \, P(x,y)$

Example

$\forall x\, \exists y\, P(x,y),\ \forall x\, \forall y\, (P(x,y) \to Q(x,y)) \not\models \exists y\, \forall x\, Q(x,y)$

model $\mathcal{M}$

- universe A: set of natural numbers
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$\mathcal{M} \models \forall x\, \exists y\, P(x,y)$

$\mathcal{M} \models \forall x\, \forall y\, (P(x,y) \to Q(x,y))$

Example

$\forall x\, \exists y\, P(x,y),\ \forall x\, \forall y\, (P(x,y) \to Q(x,y)) \not\models \exists y\, \forall x\, Q(x,y)$

model $\mathcal{M}$

- universe A: set of natural numbers
- $P^{\mathcal{M}} = Q^{\mathcal{M}} = \{(x,y) \mid x < y\}$

$\mathcal{M} \models \forall x\, \exists y\, P(x,y)$

$\mathcal{M} \models \forall x\, \forall y\, (P(x,y) \to Q(x,y))$

$\mathcal{M} \not\models \exists y\, \forall x\, Q(x,y)$

Outline

Definition

(possibly infinite) set of formulas Γ, formula ψ

- sequent $\Gamma \vdash \psi$ is valid if there exists (finite) natural deduction proof of ψ in which all premises are from Γ

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natural deduction for predicate logic is sound and complete:

$$\Gamma \vDash \psi \quad \Longleftrightarrow \quad \Gamma \vdash \psi \text{ is valid}$$

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Decision Problem

instance: set of formulas Γ, first-order formula φ

question: $\Gamma \vDash \varphi$?

Definition

(possibly infinite) set of formulas Γ, formula ψ

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Theorem (Gödel's Completeness Theorem)

natural deduction for predicate logic is sound and complete:

$$\Gamma \vDash \psi \iff \Gamma \vdash \psi \text{ is valid}$$

Decision Problem

instance: set of formulas Γ, first-order formula φ

question: $\Gamma \vDash \varphi$?

is undecidable

Definition

(possibly infinite) set of formulas Γ, formula ψ

- sequent $\Gamma \vdash \psi$ is valid if there exists (finite) natural deduction proof of ψ in which all premises are from Γ

Theorem (Gödel's Completeness Theorem)

natural deduction for predicate logic is sound and complete:

$$\Gamma \vDash \psi \quad \Longleftrightarrow \quad \Gamma \vdash \psi \text{ is valid}$$

Decision Problem (Church's Theorem)

instance: set of formulas Γ, first-order formula φ

question: $\Gamma \vDash \varphi$?

is undecidable even when $\Gamma = \varnothing$ (lecture 8)

Outline

Huth and Ryan

- Section 2.3
- Section 2.4

Huth and Ryan

- Section 2.3
- Section 2.4

Gödel's Completeness Theorem

- Wikipedia [accessed January 24, 2024]

Important Concepts

- ∀ elimination
- ∀ introduction
- ∃ elimination
- ∃ introduction
- consistency
- environment
- equality
- equality elimination
- equality introduction
- Gödel's completeness theorem
- look-up table
- model
- satisfaction relation
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- semantic entailment
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Important Concepts

- $\forall$ elimination
- $\forall$ introduction
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- consistency
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homework for April 25