



## Logic

Diana Gründlinger

Aart Middeldorp

Fabian Mitterwallner

Alexander Montag

Johannes Niederhauser

Daniel Rainer

 **articify**  
ars.uibk.ac.at

with session ID **0992 9580** for anonymous questions



# Outline

- 1. Summary of Previous Lecture**
- 2. Quantifier Equivalences**
- 3. Intermezzo**
- 4. Unification**
- 5. Intermezzo**
- 6. Skolemization**
- 7. Further Reading**

## Definitions

- ▶ **model**  $\mathcal{M}$  for pair  $(\mathcal{F}, \mathcal{P})$  with set  $\mathcal{F}$  of function symbols and set  $\mathcal{P}$  of predicate symbols consists of
  - ① non-empty set  $A$  (universe of concrete values)
  - ② function  $f^{\mathcal{M}}: A^n \rightarrow A$  for every  $n$ -ary function symbol  $f \in \mathcal{F}$
  - ③ subset  $P^{\mathcal{M}} \subseteq A^n$  for every  $n$ -ary predicate symbol  $P \in \mathcal{P}$
  - ④  $=^{\mathcal{M}}$  is identity relation on  $A$
- ▶ **environment** (look-up table) for model  $\mathcal{M} = (A, \{f^{\mathcal{M}}\}_{f \in \mathcal{F}}, \{P^{\mathcal{M}}\}_{P \in \mathcal{P}})$  is mapping  $I$  from variables to elements of  $A$
- ▶ value  $t^{\mathcal{M}, I}$  of term  $t$  in model  $\mathcal{M}$  relative to environment  $I$  is defined inductively:

$$t^{\mathcal{M}, I} = \begin{cases} I(t) & \text{if } t \text{ is variable} \\ f^{\mathcal{M}}(t_1^{\mathcal{M}, I}, \dots, t_n^{\mathcal{M}, I}) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

## Definitions

- ▶ **satisfaction** relation  $\mathcal{M} \models_I \varphi$  is defined inductively:

$$\begin{array}{l} \mathcal{M} \models_I \top \\ \mathcal{M} \not\models_I \perp \\ \mathcal{M} \models_I \varphi \iff \end{array} \left\{ \begin{array}{ll} (t_1^{M,I}, \dots, t_n^{M,I}) \in P^{\mathcal{M}} & \text{if } \varphi = P(t_1, \dots, t_n) \\ \mathcal{M} \not\models_I \psi & \text{if } \varphi = \neg \psi \\ \mathcal{M} \models_I \psi_1 \text{ and } \mathcal{M} \models_I \psi_2 & \text{if } \varphi = \psi_1 \wedge \psi_2 \\ \mathcal{M} \models_I \psi_1 \text{ or } \mathcal{M} \models_I \psi_2 & \text{if } \varphi = \psi_1 \vee \psi_2 \\ \mathcal{M} \not\models_I \psi_1 \text{ or } \mathcal{M} \models_I \psi_2 & \text{if } \varphi = \psi_1 \rightarrow \psi_2 \\ \mathcal{M} \models_{I[x \mapsto a]} \psi \text{ for all } a \in A & \text{if } \varphi = \forall x \psi \\ \mathcal{M} \models_{I[x \mapsto a]} \psi \text{ for some } a \in A & \text{if } \varphi = \exists x \psi \end{array} \right.$$

- ▶ formula  $\psi$  is **satisfiable** if  $\mathcal{M} \models_I \psi$  for some model  $\mathcal{M}$  and environment  $I$
- ▶ formula  $\psi$  is **valid** if  $\mathcal{M} \models_I \psi$  for all (appropriate) models  $\mathcal{M}$  and environments  $I$

## Definitions

(possibly infinite) set of formulas  $\Gamma$

- ▶  $\Gamma$  is **satisfiable** (**consistent**) if  $\mathcal{M} \models_I \varphi$  for all  $\varphi \in \Gamma$ , for some model  $\mathcal{M}$  and environment  $I$
- ▶  $\Gamma \models \psi$  (**semantic entailment**) if  $\mathcal{M} \models_I \psi$  whenever  $\mathcal{M} \models_I \varphi$  for all  $\varphi \in \Gamma$ , for all (appropriate) models  $\mathcal{M}$  and environments  $I$

## Definitions

- ▶ **equality introduction**

$$\frac{}{t = t} =i$$

- ▶ **equality elimination**

$$\frac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]} =e \quad \text{"replace equals by equals"}$$

provided  $t_1$  and  $t_2$  are free for  $x$  in  $\varphi$

## Definitions

### ▶ $\forall$ elimination

$$\frac{\forall x \varphi}{\varphi[t/x]} \forall e$$

provided  $t$  is free for  $x$  in  $\varphi$

### ▶ $\forall$ introduction

$$\frac{\boxed{\begin{array}{c} x_0 \\ \vdots \\ \varphi[x_0/x] \end{array}}}{\forall x \varphi} \forall i$$

where  $x_0$  is fresh variable that is used only inside box

### $\exists$ introduction

$$\frac{\varphi[t/x]}{\exists x \varphi} \exists i$$

### $\exists$ elimination

$$\frac{\exists x \varphi \quad \boxed{\begin{array}{c} x_0 \quad \varphi[x_0/x] \\ \vdots \\ \chi \end{array}}}{\chi} \exists e$$

## Definition

(possibly infinite) set of formulas  $\Gamma$ , formula  $\psi$

- ▶ **sequent**  $\Gamma \vdash \psi$  is **valid** if there exists (finite) natural deduction proof of  $\psi$  in which all premises are from  $\Gamma$

## Theorem (Gödel's Completeness Theorem)

natural deduction for predicate logic is **sound** and **complete**:

$$\Gamma \models \psi \iff \Gamma \vdash \psi \text{ is valid}$$

## Decision Problem (Church's Theorem)

instance: set of formulas  $\Gamma$ , first-order formula  $\psi$

question:  $\Gamma \models \psi$  ?

is **undecidable** even when  $\Gamma = \emptyset$



## Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, soundness and completeness, syntax, Tseitin's transformation

## Part II: Predicate Logic

natural deduction, **quantifier equivalences**, resolution, semantics, **Skolemization**, syntax, undecidability, **unification**

## Part III: Model Checking

adequacy, branching-time temporal logic, CTL\*, fairness, linear-time temporal logic, model checking algorithms, symbolic model checking

## Example

Consider the universe  $A$  consisting of the set of all humans. Given the following premises:

- 1 Every child likes sweets or is already full (or both).
- 2 If a human is sad, they no longer like sweets.
- 3 There is at least one child who is sad.

Using natural deduction, prove that **there is at least one child who is full.**

$$\forall x (C(x) \rightarrow L(x) \vee F(x)), \quad S(x) \rightarrow \neg L(x), \quad \exists x (C(x) \wedge S(x)) \not\models \exists x (C(x) \wedge F(x))$$

countermodel  $\mathcal{M}$  with look-up table  $I(x) = \text{Diana}$

- ▶ Diana, Jamie  $\in A$
- ▶ Diana  $\notin C^{\mathcal{M}}$  Diana  $\in L^{\mathcal{M}}$  Diana  $\notin S^{\mathcal{M}}$  Diana  $\notin F^{\mathcal{M}}$
- ▶ Jamie  $\in C^{\mathcal{M}}$  Jamie  $\in L^{\mathcal{M}}$  Jamie  $\in S^{\mathcal{M}}$  Jamie  $\notin F^{\mathcal{M}}$
- ▶  $\mathcal{M} \models \forall x (C(x) \rightarrow L(x) \vee F(x))$   $\mathcal{M} \models S(x) \rightarrow \neg L(x)$   $\mathcal{M} \models \exists x (C(x) \wedge S(x))$
- ▶  $\mathcal{M} \not\models \exists x (C(x) \wedge F(x))$

# Outline

1. Summary of Previous Lecture

**2. Quantifier Equivalences**

3. Intermezzo

4. Unification

5. Intermezzo

6. Skolemization

7. Further Reading

## Notation

$\varphi \dashv\vdash \psi$  denotes validity of both  $\varphi \vdash \psi$  and  $\psi \vdash \varphi$

## Theorem

$$\neg \forall x \varphi \dashv\vdash \exists x \neg \varphi$$

$$\forall x \varphi \wedge \forall x \psi \dashv\vdash \forall x (\varphi \wedge \psi)$$

$$\forall x \forall y \varphi \dashv\vdash \forall y \forall x \varphi$$

$$\neg \exists x \varphi \dashv\vdash \forall x \neg \varphi$$

$$\exists x \varphi \vee \exists x \psi \dashv\vdash \exists x (\varphi \vee \psi)$$

$$\exists x \exists y \varphi \dashv\vdash \exists y \exists x \varphi$$

if  $x$  is not free in  $\psi$  then

$$\forall x \varphi \wedge \psi \dashv\vdash \forall x (\varphi \wedge \psi)$$

$$\exists x \varphi \wedge \psi \dashv\vdash \exists x (\varphi \wedge \psi)$$

$$\psi \rightarrow \forall x \varphi \dashv\vdash \forall x (\psi \rightarrow \varphi)$$

$$\psi \rightarrow \exists x \varphi \dashv\vdash \exists x (\psi \rightarrow \varphi)$$

$$\forall x \varphi \vee \psi \dashv\vdash \forall x (\varphi \vee \psi)$$

$$\exists x \varphi \vee \psi \dashv\vdash \exists x (\varphi \vee \psi)$$

$$\exists x \varphi \rightarrow \psi \dashv\vdash \forall x (\varphi \rightarrow \psi)$$

$$\forall x \varphi \rightarrow \psi \dashv\vdash \exists x (\varphi \rightarrow \psi)$$

$\exists x \neg \varphi \vdash \neg \forall x \varphi$  is valid:

1	$\exists x \neg \varphi$	premise
2	$\forall x \varphi$	assumption
3	$x_0 \ (\neg \varphi)[x_0/x]$	assumption
4	$\neg(\varphi[x_0/x])$	identical
5	$\varphi[x_0/x]$	$\forall e$ 2
6	$\perp$	$\neg e$ 5, 4
7	$\perp$	$\exists e$ 1, 3-6
8	$\neg \forall x \varphi$	$\neg i$ 2-7

$\exists x \varphi \vee \exists x \psi \vdash \exists x (\varphi \vee \psi)$  is valid:

1	$\exists x \varphi \vee \exists x \psi$	premise
2	$\exists x \varphi$	assumption
3	$x_0 \quad \varphi[x_0/x]$	assumption
4	$\varphi[x_0/x] \vee \psi[x_0/x]$	$\vee i_1$ 3
5	$\exists x (\varphi \vee \psi)$	$\exists i$ 4
6	$\exists x (\varphi \vee \psi)$	$\exists e$ 2, 3–5
7	$\exists x \psi$	assumption
8	$x_0 \quad \psi[x_0/x]$	assumption
9	$\varphi[x_0/x] \vee \psi[x_0/x]$	$\vee i_2$ 8
10	$\exists x (\varphi \vee \psi)$	$\exists i$ 9
11	$\exists x (\varphi \vee \psi)$	$\exists e$ 7, 8–10
12	$\exists x (\varphi \vee \psi)$	$\vee e$ 1, 2–6, 7–11

$\exists x (\varphi \vee \psi) \vdash \exists x \varphi \vee \exists x \psi$  is valid:

1	$\exists x (\varphi \vee \psi)$	premise
2	$x_0 (\varphi \vee \psi)[x_0/x]$	assumption
3	$\varphi[x_0/x] \vee \psi[x_0/x]$	identical
4	$\varphi[x_0/x]$	assumption
5	$\exists x \varphi$	$\exists i$ 4
6	$\exists x \varphi \vee \exists x \psi$	$\vee i_1$ 5
7	$\psi[x_0/x]$	assumption
8	$\exists x \psi$	$\exists i$ 7
9	$\exists x \varphi \vee \exists x \psi$	$\vee i_2$ 8
10	$\exists x \varphi \vee \exists x \psi$	$\vee e$ 3, 4–6, 7–9
11	$\exists x \varphi \vee \exists x \psi$	$\exists e$ 1, 2–10

$\forall x \forall y \varphi \vdash \forall y \forall x \varphi$  is valid:

1  $\forall x \forall y \varphi$  premise

2  $y_0$

3  $x_0 (\forall y \varphi)[x_0/x] \quad \forall e 1$

4  $\forall y (\varphi[x_0/x])$  identical

5  $\varphi[x_0/x][y_0/y] \quad \forall e 4$

6  $\varphi[y_0/y][x_0/x]$  identical

7  $\forall x (\varphi[y_0/y]) \quad \forall i 3-6$

8  $(\forall x \varphi)[y_0/y]$  identical

9  $\forall y \forall x \varphi \quad \forall i 2-8$



$\exists x \exists y \varphi \vdash \exists y \exists x \varphi$  is valid:

1	$\exists x \exists y \varphi$	premise
2	$x_0 (\exists y \varphi)[x_0/x]$	assumption
3	$\exists y (\varphi[x_0/x])$	identical
4	$y_0 \varphi[x_0/x][y_0/y]$	assumption
5	$\varphi[y_0/y][x_0/x]$	identical
6	$\exists x (\varphi[y_0/y])$	$\exists i 5$
7	$(\exists x \varphi)[y_0/y]$	identical
8	$\exists y \exists x \varphi$	$\exists i 7$
9	$\exists y \exists x \varphi$	$\exists e 3, 4-8$
10	$\exists y \exists x \varphi$	$\exists e 1, 2-9$

## Proof

$\forall x \varphi \wedge \psi \vdash \forall x (\varphi \wedge \psi)$  is valid (provided  $x$  is not free in  $\psi$ ):

1	$\forall x \varphi \wedge \psi$	premise
2	$\forall x \varphi$	$\wedge e_1$ 1
3	$\psi$	$\wedge e_2$ 1
4	$x_0 \quad \varphi[x_0/x]$	$\forall e$ 2
5	$\varphi[x_0/x] \wedge \psi$	$\wedge i$ 4, 3
6	$(\varphi \wedge \psi)[x_0/x]$	identical
7	$\forall x (\varphi \wedge \psi)$	$\forall i$ 4-6

## Remark

freeness condition is essential:  $\forall x P(x) \wedge Q(x) \not\equiv \forall x (P(x) \wedge Q(x))$

- ▶ model  $\mathcal{M}$  with universe  $\{0, 1\}$ ,  $P^{\mathcal{M}} = \{0, 1\}$ ,  $Q^{\mathcal{M}} = \{0\}$  and environment  $I(x) = 0$
- ▶  $\mathcal{M} \models \forall x P(x) \wedge Q(x)$  and  $\mathcal{M} \not\models \forall x (P(x) \wedge Q(x))$

$\forall x (\varphi \wedge \psi) \vdash \forall x \varphi \wedge \psi$  is valid (provided  $x$  is not free in  $\psi$ ):

1	$\forall x (\varphi \wedge \psi)$	premise
2	$x_0 (\varphi \wedge \psi)[x_0/x]$	$\forall e$ 1
3	$\varphi[x_0/x] \wedge \psi$	identical
4	$\psi$	$\wedge e_2$ 3
5	$\varphi[x_0/x]$	$\wedge e_1$ 3
6	$\forall x \varphi$	$\forall i$ 2–5
7	$(\varphi \wedge \psi)[x/x]$	$\forall e$ 1
8	$\varphi \wedge \psi$	identical
9	$\psi$	$\wedge e_2$ 8
10	$\forall x \varphi \wedge \psi$	$\wedge i$ 6,9

$\forall x \varphi \vee \psi \vdash \forall x (\varphi \vee \psi)$  is valid (provided  $x$  is not free in  $\psi$ ):

1	$\forall x \varphi \vee \psi$	premise
2	$\forall x \varphi$	assumption
3	$x_0 \quad \varphi[x_0/x]$	$\forall e \ 2$
4	$\varphi[x_0/x] \vee \psi[x_0/x]$	$\vee i_1 \ 3$
5	$(\varphi \vee \psi)[x_0/x]$	identical
6	$\forall x (\varphi \vee \psi)$	$\forall i \ 3-5$
7	$\psi$	assumption
8	$x_0 \quad \varphi[x_0/x] \vee \psi$	$\vee i_2 \ 7$
9	$(\varphi \vee \psi)[x_0/x]$	identical
10	$\forall x (\varphi \vee \psi)$	$\forall i \ 8-9$
11	$\forall x (\varphi \vee \psi)$	$\vee e \ 1, 2-6, 7-10$

$\forall x (\varphi \vee \psi) \vdash \forall x \varphi \vee \psi$  is valid (provided  $x$  is not free in  $\psi$ ):

1	$\forall x (\varphi \vee \psi)$	premise	5
2	$\psi \vee \neg\psi$	LEM	6
3	$\psi$	assumption	7
4	$\forall x \varphi \vee \psi$	$\forall i_1$ 3	8
			9
			10
			11
			12
			13
			14
			15

	$\neg\psi$	assumption
$x_0$	$(\varphi \vee \psi)[x_0/x]$	$\forall e$ 1
	$\varphi[x_0/x] \vee \psi$	identical
	$\varphi[x_0/x]$	assumption
	$\psi$	assumption
	$\perp$	$\neg e$ 9, 5
	$\varphi[x_0/x]$	$\perp e$ 10
	$\varphi[x_0/x]$	$\forall e$ 7, 8–8, 9–11
	$\forall x \varphi$	$\forall i$ 6–12
	$\forall x \varphi \vee \psi$	$\forall i_1$ 13
	$\forall x \varphi \vee \psi$	$\forall e$ 2, 3–4, 5–14

$\forall x (\psi \rightarrow \varphi) \vdash \psi \rightarrow \forall x \varphi$  is valid (provided  $x$  is not free in  $\psi$ ):

1	$\forall x (\psi \rightarrow \varphi)$	premise
2	$\psi$	assumption
3	$x_0 (\psi \rightarrow \varphi)[x_0/x]$	$\forall e$ 1
4	$\psi \rightarrow \varphi[x_0/x]$	identical
5	$\varphi[x_0/x]$	$\rightarrow e$ 4, 2
6	$\forall x \varphi$	$\forall i$ 3–5
7	$\psi \rightarrow \forall x \varphi$	$\rightarrow i$ 2–6

$\psi \rightarrow \forall x \varphi \vdash \forall x (\psi \rightarrow \varphi)$  is valid (provided  $x$  is not free in  $\psi$ ):

1	$\psi \rightarrow \forall x \varphi$	premise
2	$x_0$	
3	$\psi$	assumption
4	$\forall x \varphi$	$\rightarrow e$ 1, 3
5	$\varphi[x_0/x]$	$\forall e$ 4
6	$\psi \rightarrow \varphi[x_0/x]$	$\rightarrow i$ 3–5
7	$(\psi \rightarrow \varphi)[x_0/x]$	identical
8	$\forall x (\psi \rightarrow \varphi)$	$\forall i$ 2–7

# Outline

1. Summary of Previous Lecture
2. Quantifier Equivalences
- 3. Intermezzo**
4. Unification
5. Intermezzo
6. Skolemization
7. Further Reading



## Question

Which of the following formulas are equivalent to the formula

$$\neg \exists x \forall y \neg \exists z \varphi \rightarrow \forall z \psi$$

if  $z$  is free in  $\varphi$  and  $\psi$ , and  $x$  and  $y$  are not free in  $\psi$ ?

- A**  $\exists x \forall y \neg (\exists z \varphi \rightarrow \forall z \psi)$
- B**  $\exists x \forall y (\exists z \varphi \rightarrow \forall z \psi)$
- C**  $\exists x \forall y \forall z (\varphi \rightarrow \psi)$
- D**  $\exists x \forall y \forall z (\varphi \rightarrow \forall z \psi)$
- E**  $\exists x \forall y \exists z (\varphi \rightarrow \psi)$



# Outline

1. Summary of Previous Lecture
2. Quantifier Equivalences
3. Intermezzo
- 4. Unification**
5. Intermezzo
6. Skolemization
7. Further Reading

## Definitions

- ▶ **substitution** is set of variable bindings  $\theta = \{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$  with pairwise different variables  $x_1, \dots, x_n$  and terms  $t_1, \dots, t_n$
- ▶ given substitution  $\theta = \{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$  and expression  $E$ , **instance**  $E\theta$  of  $E$  is obtained by simultaneously replacing each occurrence of  $x_i$  in  $E$  by  $t_i$
- ▶ **composition** of substitutions  $\theta = \{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$  and  $\sigma = \{y_1 \mapsto s_1, \dots, y_k \mapsto s_k\}$  is substitution  $\theta\sigma = \{x_1 \mapsto t_1\sigma, \dots, x_n \mapsto t_n\sigma\} \cup \{y_i \mapsto s_i \mid y_i \neq x_j \text{ for all } 1 \leq j \leq n\}$

## Example

$$\theta = \{x \mapsto g(y, z), y \mapsto a\}$$

$$E = P(f(y), x, y)$$

$$\sigma = \{x \mapsto f(y), z \mapsto f(x)\}$$

$$E\theta = P(f(a), g(y, z), a)$$

$$\theta\sigma = \{x \mapsto g(y, f(x)), y \mapsto a, z \mapsto f(x)\}$$

$$E\sigma = P(f(y), f(y), y)$$

$$\sigma\theta = \{x \mapsto f(a), z \mapsto f(g(y, z)), y \mapsto a\}$$

## Lemma

composition of substitutions is associative:  $(\rho\sigma)\tau = \rho(\sigma\tau)$

## Definitions

- ▶ substitution  $\theta$  is **at least as general** as substitution  $\sigma$  if  $\theta\mu = \sigma$  for some substitution  $\mu$
- ▶ **unifier** of terms  $s$  and  $t$  is substitution  $\theta$  such that  $s\theta = t\theta$
- ▶ **most general unifier (mgu)** is at least as general as any other unifier

## Example

terms  $f(x, g(y), x)$  and  $f(z, g(u), h(u))$  are unifiable:

$$\{x \mapsto h(a), y \mapsto a, z \mapsto h(a), u \mapsto a\} \qquad \{u \mapsto a\}$$

unifiers  $\{x \mapsto h(u), y \mapsto u, z \mapsto h(u)\}$  **mgu**

$$\{x \mapsto h(g(u)), y \mapsto g(u), z \mapsto h(g(u)), u \mapsto g(u)\} \qquad \{u \mapsto g(u)\}$$

## Theorem

unifiable terms have mgu which can be computed by unification algorithm

## Unification Algorithm

**d** decomposition

$$\frac{E_1, f(s_1, \dots, s_n) \approx f(t_1, \dots, t_n), E_2}{E_1, s_1 \approx t_1, \dots, s_n \approx t_n, E_2}$$

**t** removal of trivial equations

$$\frac{E_1, t \approx t, E_2}{E_1, E_2}$$

**v** variable elimination

$$\frac{E_1, x \approx t, E_2}{(E_1, E_2)\{x \mapsto t\}} \quad \text{and} \quad \frac{E_1, t \approx x, E_2}{(E_1, E_2)\{x \mapsto t\}}$$

if  $x$  does not occur in  $t$  (**occurs check**)

## Example

$$f(x, g(y), x) \approx f(z, g(u), h(u))$$

d  $\Downarrow$

$$x \approx z, g(y) \approx g(u), x \approx h(u)$$

v  $\Downarrow$   $\{x \mapsto z\}$

$$g(y) \approx g(u), z \approx h(u)$$

d  $\Downarrow$

mgu  $\{x \mapsto h(u), y \mapsto u, z \mapsto h(u)\}$

$$y \approx u, z \approx h(u)$$

v  $\Downarrow$   $\{y \mapsto u\}$

$$z \approx h(u)$$

v  $\Downarrow$   $\{z \mapsto h(u)\}$

□

## Theorem

- ▶ there are no infinite derivations

$$U \Rightarrow_{\theta_1} V \Rightarrow_{\theta_2} W \Rightarrow_{\theta_3} \dots$$

- ▶ if  $s$  and  $t$  are unifiable then for every maximal derivation

$$s \approx t \Rightarrow_{\theta_1} E_1 \Rightarrow_{\theta_2} E_2 \Rightarrow_{\theta_3} \dots \Rightarrow_{\theta_n} E_n$$

- ▶  $E_n = \square$

- ▶  $\theta_1 \theta_2 \theta_3 \dots \theta_n$  is mgu of  $s$  and  $t$

## Optional Failure Rules

$$\frac{E_1, f(s_1, \dots, s_n) \approx g(t_1, \dots, t_m), E_2}{\perp}$$

$$\frac{E_1, x \approx t, E_2}{\perp} \quad \frac{E_1, t \approx x, E_2}{\perp}$$

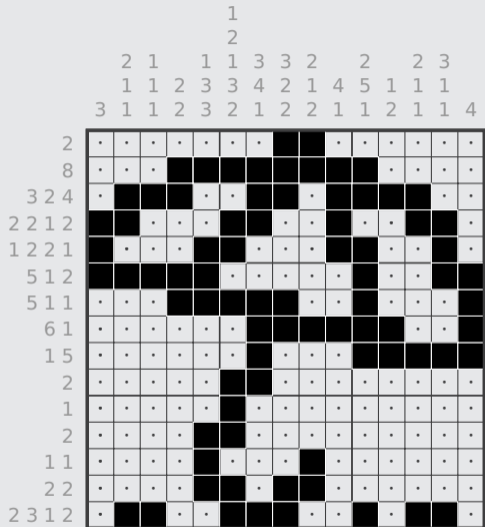
if  $x$  occurs in  $t$  and  $x \neq t$

# Outline

1. Summary of Previous Lecture
2. Quantifier Equivalences
3. Intermezzo
4. Unification
- 5. Intermezzo**
6. Skolemization
7. Further Reading



# Example (Picture Logic)



# Outline

1. Summary of Previous Lecture
2. Quantifier Equivalences
3. Intermezzo
4. Unification
5. Intermezzo
- 6. Skolemization**
7. Further Reading

## Definitions

- ▶ **prenex normal form** is predicate logic formula

$$Q_1 x_1 Q_2 x_2 \dots Q_n x_n \varphi$$

with  $Q_i \in \{\forall, \exists\}$  and  $\varphi$  quantifier-free

- ▶ **Skolem normal form** is closed (no free variables) prenex normal form

$$\forall x_1 \forall x_2 \dots \forall x_n \varphi$$

with  $\varphi$  quantifier-free CNF

## Example

$$\forall x \forall y ((P(f(x)) \vee \neg P(g(y)) \vee Q(g(y))) \wedge (\neg Q(g(y)) \vee \neg P(g(y)) \vee Q(g(x))))$$

**clausal form**  $\{ \{P(f(x)), \neg P(g(y)), Q(g(y))\}, \{\neg Q(g(y)), \neg P(g(y)), Q(g(x))\} \}$

## Theorem

for every formula  $\varphi$  there exists prenex normal form  $\psi$  such that  $\varphi \equiv \psi$

## Proof

- ① rename all bound variables such that all variables in quantifications are distinct
- ② push logical connectives through quantifiers:

$$\neg \forall x \varphi \equiv \exists x \neg \varphi$$

$$\forall x \varphi \wedge \psi \equiv \forall x (\varphi \wedge \psi)$$

$$\exists x \varphi \wedge \psi \equiv \exists x (\varphi \wedge \psi)$$

$$\forall x \varphi \vee \psi \equiv \forall x (\varphi \vee \psi)$$

$$\exists x \varphi \vee \psi \equiv \exists x (\varphi \vee \psi)$$

$$\forall x \varphi \rightarrow \psi \equiv \exists x (\varphi \rightarrow \psi)$$

$$\exists x \varphi \rightarrow \psi \equiv \forall x (\varphi \rightarrow \psi)$$

$$\neg \exists x \varphi \equiv \forall x \neg \varphi$$

$$\varphi \wedge \forall x \psi \equiv \forall x (\varphi \wedge \psi)$$

$$\varphi \wedge \exists x \psi \equiv \exists x (\varphi \wedge \psi)$$

$$\varphi \vee \forall x \psi \equiv \forall x (\varphi \vee \psi)$$

$$\varphi \vee \exists x \psi \equiv \exists x (\varphi \vee \psi)$$

$$\varphi \rightarrow \forall x \psi \equiv \forall x (\varphi \rightarrow \psi)$$

$$\varphi \rightarrow \exists x \psi \equiv \exists x (\varphi \rightarrow \psi)$$

## Theorem

for every sentence  $\varphi$  there exists Skolem normal form  $\psi$  such that  $\varphi \approx \psi$

## Proof (Skolemization)

- ① transform  $\varphi$  into closed prenex normal form  $Q_1 x_1 Q_2 x_2 \dots Q_n x_n \chi$  with  $\chi$  in CNF
- ② repeatedly replace  $\forall x_1 \dots \forall x_{i-1} \exists x_i Q_{i+1} x_{i+1} \dots Q_n x_n \psi$  by

$$\forall x_1 \dots \forall x_{i-1} Q_{i+1} x_{i+1} \dots Q_n x_n \psi[f(x_1, \dots, x_{i-1})/x_i]$$

where  $f$  is new function symbol of arity  $i - 1$  (if  $i = 1$  then  $f$  is constant)

## Remark

**unification** and **Skolemization** are required to extend resolution from propositional logic to predicate logic

## Examples

$$① \quad \forall z \exists x \exists y ((P(x) \vee \neg P(y) \vee Q(z)) \wedge (\neg Q(x) \vee \neg P(y) \vee Q(z)))$$

$$\approx \{x \mapsto f(z)\}$$

$$\forall z \exists y ((P(f(z)) \vee \neg P(y) \vee Q(z)) \wedge (\neg Q(f(z)) \vee \neg P(y) \vee Q(z)))$$

$$\approx \{y \mapsto g(z)\}$$

$$\forall z ((P(f(z)) \vee \neg P(g(z)) \vee Q(z)) \wedge (\neg Q(f(z)) \vee \neg P(g(z)) \vee Q(z)))$$

$$② \quad \forall x \exists y (\exists z P(z) \wedge (\exists u Q(x, u) \rightarrow \exists v Q(y, v)))$$

$\equiv$

$$\forall x \exists y \exists z \forall u \exists v (P(z) \wedge (\neg Q(x, u) \vee Q(y, v)))$$

$$\approx \{y \mapsto f(x)\} \{z \mapsto g(x)\}$$

$$\forall x \forall u \exists v (P(g(x)) \wedge (\neg Q(x, u) \vee Q(f(x), v)))$$

$$\approx \{v \mapsto h(x, u)\}$$

$$\forall x \forall u (P(g(x)) \wedge (\neg Q(x, u) \vee Q(f(x), h(x, u))))$$

# Outline

1. Summary of Previous Lecture
2. Quantifier Equivalences
3. Intermezzo
4. Unification
5. Intermezzo
6. Skolemization
- 7. Further Reading**

## Huth and Ryan

- ▶ Section 2.3

## Unification

- ▶ Wikipedia

[accessed January 25, 2024]

## Skolemization

- ▶ Wikipedia

[accessed January 25, 2024]



## Important Concepts

- ▶ at least as general
- ▶ composition
- ▶ decomposition
- ▶ instance
- ▶ most general unifier
- ▶ occurs check
- ▶ prenex normal form
- ▶ Skolem normal form
- ▶ Skolemization
- ▶ quantifier equivalences
- ▶ removal of trivial equations
- ▶ substitution
- ▶ unification algorithm
- ▶ unifier
- ▶ variable elimination

homework for May 2