SS 2024 lecture 7

Logic

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Outline

Definitions

- model $\mathcal{M}$ for pair $(\mathcal{F}, \mathcal{P})$ with set $\mathcal{F}$ of function symbols and set $\mathcal{P}$ of predicate symbols consists of
  1. non-empty set A (universe of concrete values)
  2. function $f^{\mathcal{M}} \colon A^n \to A$ for every n-ary function symbol $f \in \mathcal{F}$
  3. subset $P^{\mathcal{M}} \subseteq A^n$ for every n-ary predicate symbol $P \in \mathcal{P}$
  4. $=^{\mathcal{M}}$ is identity relation on A
- environment (look-up table) for model $\mathcal{M} = (A, \{f^{\mathcal{M}}\}_{f \in \mathcal{F}}, \{P^{\mathcal{M}}\}_{P \in \mathcal{P}})$ is mapping l from variables to elements of A
- value $t^{\mathcal{M},l}$ of term t in model $\mathcal{M}$ relative to environment l is defined inductively:

$$t^{\mathcal{M},l} = \begin{cases} l(t) & \text{if } t \text{ is variable} \\ f^{\mathcal{M}}(t_1^{\mathcal{M},l}, \dots, t_n^{\mathcal{M},l}) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

Definitions

- satisfaction relation $\mathcal{M} \vDash_l \varphi$ is defined inductively:

$$\mathcal{M} \vDash_l \top \qquad \mathcal{M} \nvDash_l \bot$$

$$\mathcal{M} \vDash_l \varphi \iff \begin{cases} (t_1^{\mathcal{M},l}, \dots, t_n^{\mathcal{M},l}) \in P^{\mathcal{M}} & \text{if } \varphi = P(t_1, \dots, t_n) \\ \mathcal{M} \nvDash_l \psi & \text{if } \varphi = \neg\psi \\ \mathcal{M} \vDash_l \psi_1 \text{ and } \mathcal{M} \vDash_l \psi_2 & \text{if } \varphi = \psi_1 \wedge \psi_2 \\ \mathcal{M} \vDash_l \psi_1 \text{ or } \mathcal{M} \vDash_l \psi_2 & \text{if } \varphi = \psi_1 \vee \psi_2 \\ \mathcal{M} \nvDash_l \psi_1 \text{ or } \mathcal{M} \vDash_l \psi_2 & \text{if } \varphi = \psi_1 \rightarrow \psi_2 \\ \mathcal{M} \vDash_{l[x \mapsto a]} \psi \text{ for all } a \in A & \text{if } \varphi = \forall x\, \psi \\ \mathcal{M} \vDash_{l[x \mapsto a]} \psi \text{ for some } a \in A & \text{if } \varphi = \exists x\, \psi \end{cases}$$

- formula ψ is satisfiable if $\mathcal{M} \vDash_l \psi$ for some model $\mathcal{M}$ and environment l
- formula ψ is valid if $\mathcal{M} \vDash_l \psi$ for all (appropriate) models $\mathcal{M}$ and environments l

Definitions

(possibly infinite) set of formulas Γ

- Γ is satisfiable (consistent) if $\mathcal{M} \vDash_l \varphi$ for all $\varphi \in \Gamma$, for some model $\mathcal{M}$ and environment l
- $\Gamma \vDash \psi$ (semantic entailment) if $\mathcal{M} \vDash_l \psi$ whenever $\mathcal{M} \vDash_l \varphi$ for all $\varphi \in \Gamma$, for all (appropriate) models $\mathcal{M}$ and environments l

Definitions

- equality introduction $\dfrac{}{t = t}\ {=}\mathrm{i}$
- equality elimination $\dfrac{t_1 = t_2 \qquad \varphi[t_1/x]}{\varphi[t_2/x]}\ {=}\mathrm{e}$ "replace equals by equals"

  provided t_1 and t_2 are free for x in φ

Definitions

- $\forall$ elimination

$$\frac{\forall x\, \varphi}{\varphi[t/x]}\ \forall\mathrm{e}$$

provided t is free for x in φ

- $\exists$ introduction

$$\frac{\varphi[t/x]}{\exists x\, \varphi}\ \exists\mathrm{i}$$

- $\forall$ introduction

$$\frac{\boxed{\begin{array}{l} x_0 \\ \quad\vdots \\ \quad \varphi[x_0/x] \end{array}}}{\forall x\, \varphi}\ \forall\mathrm{i}$$

- $\exists$ elimination

$$\frac{\exists x\, \varphi \qquad \boxed{\begin{array}{l} x_0 \quad \varphi[x_0/x] \\ \quad\vdots \\ \quad \chi \end{array}}}{\chi}\ \exists\mathrm{e}$$

where x_0 is fresh variable that is used only inside box

Definition

(possibly infinite) set of formulas Γ, formula ψ

- sequent $\Gamma \vdash \psi$ is valid if there exists (finite) natural deduction proof of ψ in which all premises are from Γ

Theorem (Gödel's Completeness Theorem)

natural deduction for predicate logic is sound and complete:

$$\Gamma \vDash \psi \iff \Gamma \vdash \psi \text{ is valid}$$

Decision Problem (Church's Theorem)

instance: set of formulas Γ, first-order formula ψ

question: $\Gamma \vDash \psi$?

is undecidable even when $\Gamma = \varnothing$

Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, soundness and completeness, syntax, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching-time temporal logic, CTL*, fairness, linear-time temporal logic, model checking algorithms, symbolic model checking

Example

Consider the universe A consisting of the set of all humans. Given the following premises:

1. Every child likes sweets or is already full (or both).
2. If a human is sad, they no longer like sweets.
3. There is at least one child who is sad.

Using natural deduction, prove that there is at least one child who is full.

$$\forall x\,(C(x) \to L(x) \vee F(x)), \quad S(x) \to \neg L(x)\ ,\ \exists x\,(C(x) \wedge S(x)) \nvDash \exists x\,(C(x) \wedge F(x))$$

countermodel $\mathcal{M}$ with look-up table $l(x) = \text{Diana}$

- Diana, Jamie $\in A$
- Diana $\notin C^{\mathcal{M}}$ Diana $\in L^{\mathcal{M}}$ Diana $\notin S^{\mathcal{M}}$ Diana $\notin F^{\mathcal{M}}$
- Jamie $\in C^{\mathcal{M}}$ Jamie $\in L^{\mathcal{M}}$ Jamie $\in S^{\mathcal{M}}$ Jamie $\notin F^{\mathcal{M}}$
- $\mathcal{M} \vDash_l \forall x\,(C(x) \to L(x) \vee F(x))$ $\quad \mathcal{M} \vDash_l S(x) \to \neg L(x)$ $\quad \mathcal{M} \vDash_l \exists x\,(C(x) \wedge S(x))$
- $\mathcal{M} \nvDash_l \exists x\,(C(x) \wedge F(x))$

Outline

Notation

$\varphi \dashv\vdash \psi$ denotes validity of both $\varphi \vdash \psi$ and $\psi \vdash \varphi$

Theorem

$$\neg\forall x\,\varphi \dashv\vdash \exists x\,\neg\varphi \qquad \neg\exists x\,\varphi \dashv\vdash \forall x\,\neg\varphi$$

$$\forall x\,\varphi \wedge \forall x\,\psi \dashv\vdash \forall x\,(\varphi \wedge \psi) \qquad \exists x\,\varphi \vee \exists x\,\psi \dashv\vdash \exists x\,(\varphi \vee \psi)$$

$$\forall x\,\forall y\,\varphi \dashv\vdash \forall y\,\forall x\,\varphi \qquad \exists x\,\exists y\,\varphi \dashv\vdash \exists y\,\exists x\,\varphi$$

if x is not free in ψ then

$$\forall x\,\varphi \wedge \psi \dashv\vdash \forall x\,(\varphi \wedge \psi) \qquad \forall x\,\varphi \vee \psi \dashv\vdash \forall x\,(\varphi \vee \psi)$$

$$\exists x\,\varphi \wedge \psi \dashv\vdash \exists x\,(\varphi \wedge \psi) \qquad \exists x\,\varphi \vee \psi \dashv\vdash \exists x\,(\varphi \vee \psi)$$

$$\psi \to \forall x\,\varphi \dashv\vdash \forall x\,(\psi \to \varphi) \qquad \exists x\,\varphi \to \psi \dashv\vdash \forall x\,(\varphi \to \psi)$$

$$\psi \to \exists x\,\varphi \dashv\vdash \exists x\,(\psi \to \varphi) \qquad \forall x\,\varphi \to \psi \dashv\vdash \exists x\,(\varphi \to \psi)$$

Proof

$\exists x \neg\varphi \vdash \neg\forall x\, \varphi$ is valid:

1		$\exists x \neg\varphi$	premise
2		$\forall x\, \varphi$	assumption
3	x_0	$(\neg\varphi)[x_0/x]$	assumption
4		$\neg(\varphi[x_0/x])$	identical
5		$\varphi[x_0/x]$	$\forall$e 2
6		$\bot$	$\neg$e 5, 4
7		$\bot$	$\exists$e 1, 3–6
8		$\neg\forall x\, \varphi$	$\neg$i 2–7

Proof

$\exists x\, \varphi \vee \exists x\, \psi \vdash \exists x\, (\varphi \vee \psi)$ is valid:

1		$\exists x\, \varphi \vee \exists x\, \psi$	premise
2		$\exists x\, \varphi$	assumption
3	x_0	$\varphi[x_0/x]$	assumption
4		$\varphi[x_0/x] \vee \psi[x_0/x]$	$\vee$i$_1$ 3
5		$\exists x\, (\varphi \vee \psi)$	$\exists$i 4
6		$\exists x\, (\varphi \vee \psi)$	$\exists$e 2, 3–5
7		$\exists x\, \psi$	assumption
8	x_0	$\psi[x_0/x]$	assumption
9		$\varphi[x_0/x] \vee \psi[x_0/x]$	$\vee$i$_2$ 8
10		$\exists x\, (\varphi \vee \psi)$	$\exists$i 9
11		$\exists x\, (\varphi \vee \psi)$	$\exists$e 7, 8–10
12		$\exists x\, (\varphi \vee \psi)$	$\vee$e 1, 2–6, 7–11

Proof

$\exists x\, (\varphi \vee \psi) \vdash \exists x\, \varphi \vee \exists x\, \psi$ is valid:

1		$\exists x\, (\varphi \vee \psi)$	premise
2	x_0	$(\varphi \vee \psi)[x_0/x]$	assumption
3		$\varphi[x_0/x] \vee \psi[x_0/x]$	identical
4		$\varphi[x_0/x]$	assumption
5		$\exists x\, \varphi$	$\exists$i 4
6		$\exists x\, \varphi \vee \exists x\, \psi$	$\vee$i$_1$ 5
7		$\psi[x_0/x]$	assumption
8		$\exists x\, \psi$	$\exists$i 7
9		$\exists x\, \varphi \vee \exists x\, \psi$	$\vee$i$_2$ 8
10		$\exists x\, \varphi \vee \exists x\, \psi$	$\vee$e 3, 4–6, 7–9
11		$\exists x\, \varphi \vee \exists x\, \psi$	$\exists$e 1, 2–10

Proof

$\forall x\, \forall y\, \varphi \vdash \forall y\, \forall x\, \varphi$ is valid:

1		$\forall x\, \forall y\, \varphi$	premise
2	y_0		
3	x_0	$(\forall y\, \varphi)[x_0/x]$	$\forall$e 1
4		$\forall y\, (\varphi[x_0/x])$	identical
5		$\varphi[x_0/x][y_0/y]$	$\forall$e 4
6		$\varphi[y_0/y][x_0/x]$	identical
7		$\forall x\, (\varphi[y_0/y])$	$\forall$i 3–6
8		$(\forall x\, \varphi)[y_0/y]$	identical
9		$\forall y\, \forall x\, \varphi$	$\forall$i 2–8

Proof

$\exists x \, \exists y \, \varphi \vdash \exists y \, \exists x \, \varphi$ is valid:

1		$\exists x \, \exists y \, \varphi$	premise
2	x_0	$(\exists y \, \varphi)[x_0/x]$	assumption
3		$\exists y \, (\varphi[x_0/x])$	identical
4	y_0	$\varphi[x_0/x][y_0/y]$	assumption
5		$\varphi[y_0/y][x_0/x]$	identical
6		$\exists x \, (\varphi[y_0/y])$	$\exists$i 5
7		$(\exists x \, \varphi)[y_0/y]$	identical
8		$\exists y \, \exists x \, \varphi$	$\exists$i 7
9		$\exists y \, \exists x \, \varphi$	$\exists$e 3, 4–8
10		$\exists y \, \exists x \, \varphi$	$\exists$e 1, 2–9

Proof

$\forall x \, \varphi \wedge \psi \vdash \forall x \, (\varphi \wedge \psi)$ is valid (provided x is not free in ψ):

1		$\forall x \, \varphi \wedge \psi$	premise
2		$\forall x \, \varphi$	$\wedge e_1$ 1
3		ψ	$\wedge e_2$ 1
4	x_0	$\varphi[x_0/x]$	$\forall$e 2
5		$\varphi[x_0/x] \wedge \psi$	$\wedge$i 4, 3
6		$(\varphi \wedge \psi)[x_0/x]$	identical
7		$\forall x \, (\varphi \wedge \psi)$	$\forall$i 4–6

Remark

freeness condition is essential: $\forall x \, P(x) \wedge Q(x) \nvDash \forall x \, (P(x) \wedge Q(x))$

- model $\mathcal{M}$ with universe $\{0, 1\}$, $P^{\mathcal{M}} = \{0, 1\}$, $Q^{\mathcal{M}} = \{0\}$ and environment $l(x) = 0$
- $\mathcal{M} \vDash_l \forall x \, P(x) \wedge Q(x)$ and $\mathcal{M} \nvDash_l \forall x \, (P(x) \wedge Q(x))$

Proof

$\forall x \, (\varphi \wedge \psi) \vdash \forall x \, \varphi \wedge \psi$ is valid (provided x is not free in ψ):

1		$\forall x \, (\varphi \wedge \psi)$	premise
2	x_0	$(\varphi \wedge \psi)[x_0/x]$	$\forall$e 1
3		$\varphi[x_0/x] \wedge \psi$	identical
4		ψ	$\wedge e_2$ 3
5		$\varphi[x_0/x]$	$\wedge e_1$ 3
6		$\forall x \, \varphi$	$\forall$i 2–5
7		$(\varphi \wedge \psi)[x/x]$	$\forall$e 1
8		$\varphi \wedge \psi$	identical
9		ψ	$\wedge e_2$ 8
10		$\forall x \, \varphi \wedge \psi$	$\wedge$i 6, 9

Proof

$\forall x \, \varphi \vee \psi \vdash \forall x \, (\varphi \vee \psi)$ is valid (provided x is not free in ψ):

1		$\forall x \, \varphi \vee \psi$	premise
2		$\forall x \, \varphi$	assumption
3	x_0	$\varphi[x_0/x]$	$\forall$e 2
4		$\varphi[x_0/x] \vee \psi[x_0/x]$	$\vee i_1$ 3
5		$(\varphi \vee \psi)[x_0/x]$	identical
6		$\forall x \, (\varphi \vee \psi)$	$\forall$i 3–5
7		ψ	assumption
8	x_0	$\varphi[x_0/x] \vee \psi$	$\vee i_2$ 7
9		$(\varphi \vee \psi)[x_0/x]$	identical
10		$\forall x \, (\varphi \vee \psi)$	$\forall$i 8–9
11		$\forall x \, (\varphi \vee \psi)$	$\vee$e 1, 2–6, 7–10

Proof

$\forall x\,(\varphi \vee \psi) \vdash \forall x\,\varphi \vee \psi$ is valid (provided x is not free in ψ):

1		$\forall x\,(\varphi \vee \psi)$	premise
2		$\psi \vee \neg\psi$	LEM
3		ψ	assumption
4		$\forall x\,\varphi \vee \psi$	$\vee i_1$ 3
5		$\neg\psi$	assumption
6	x_0	$(\varphi \vee \psi)[x_0/x]$	$\forall e$ 1
7		$\varphi[x_0/x] \vee \psi$	identical
8		$\varphi[x_0/x]$	assumption
9		ψ	assumption
10		$\bot$	$\neg e$ 9, 5
11		$\varphi[x_0/x]$	$\bot e$ 10
12		$\varphi[x_0/x]$	$\vee e$ 7, 8–8, 9–11
13		$\forall x\,\varphi$	$\forall i$ 6–12
14		$\forall x\,\varphi \vee \psi$	$\vee i_1$ 13
15		$\forall x\,\varphi \vee \psi$	$\vee e$ 2, 3–4, 5–14

Proof

$\forall x\,(\psi \to \varphi) \vdash \psi \to \forall x\,\varphi$ is valid (provided x is not free in ψ):

1		$\forall x\,(\psi \to \varphi)$	premise
2		ψ	assumption
3	x_0	$(\psi \to \varphi)[x_0/x]$	$\forall e$ 1
4		$\psi \to \varphi[x_0/x]$	identical
5		$\varphi[x_0/x]$	$\to e$ 4, 2
6		$\forall x\,\varphi$	$\forall i$ 3–5
7		$\psi \to \forall x\,\varphi$	$\to i$ 2–6

Proof

$\psi \to \forall x\,\varphi \vdash \forall x\,(\psi \to \varphi)$ is valid (provided x is not free in ψ):

1		$\psi \to \forall x\,\varphi$	premise
2	x_0		
3		ψ	assumption
4		$\forall x\,\varphi$	$\to e$ 1, 3
5		$\varphi[x_0/x]$	$\forall e$ 4
6		$\psi \to \varphi[x_0/x]$	$\to i$ 3–5
7		$(\psi \to \varphi)[x_0/x]$	identical
8		$\forall x\,(\psi \to \varphi)$	$\forall i$ 2–7

Outline

1. Summary of Previous Lecture
2. Quantifier Equivalences
3. **Intermezzo**
4. Unification
5. Intermezzo
6. Skolemization
7. Further Reading

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Question

Which of the following formulas are equivalent to the formula

$$\neg \exists x \, \forall y \, \neg \exists z \, \varphi \to \forall z \, \psi$$

if z is free in φ and ψ, and x and y are not free in ψ ?

A $\exists x \, \forall y \, \neg(\exists z \, \varphi \to \forall z \, \psi)$

B $\exists x \, \forall y \, (\exists z \, \varphi \to \forall z \, \psi)$

C $\exists x \, \forall y \, \forall z \, (\varphi \to \psi)$

D $\exists x \, \forall y \, \forall z \, (\varphi \to \forall z \, \psi)$

E $\exists x \, \forall y \, \exists z \, (\varphi \to \psi)$

Outline

Definitions

- substitution is set of variable bindings $\theta = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ with pairwise different variables $x_1, \ldots, x_n$ and terms $t_1, \ldots, t_n$
- given substitution $\theta = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ and expression E, instance $E\theta$ of E is obtained by simultaneously replacing each occurrence of x_i in E by t_i
- composition of substitutions $\theta = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ and $\sigma = \{y_1 \mapsto s_1, \ldots, y_k \mapsto s_k\}$ is substitution $\theta\sigma = \{x_1 \mapsto t_1\sigma, \ldots, x_n \mapsto t_n\sigma\} \cup \{y_i \mapsto s_i \mid y_i \neq x_j \text{ for all } 1 \leqslant j \leqslant n\}$

Example

$$\theta = \{x \mapsto g(y,z),\ y \mapsto a\} \qquad E = P(f(y),x,y)$$

$$\sigma = \{x \mapsto f(y),\ z \mapsto f(x)\} \qquad E\theta = P(f(a),g(y,z),a)$$

$$\theta\sigma = \{x \mapsto g(y,f(x)),\ y \mapsto a,\ z \mapsto f(x)\} \qquad E\sigma = P(f(y),f(y),y)$$

$$\sigma\theta = \{x \mapsto f(a),\ z \mapsto f(g(y,z)),\ y \mapsto a\}$$

Lemma

composition of substitutions is associative: $(\rho\sigma)\tau = \rho(\sigma\tau)$

Definitions

- substitution θ is at least as general as substitution σ if $\theta\mu = \sigma$ for some substitution μ
- unifier of terms s and t is substitution θ such that $s\theta = t\theta$
- most general unifier (mgu) is at least as general as any other unifier

Example

terms $f(x,g(y),x)$ and $f(z,g(u),h(u))$ are unifiable:

	$\{x \mapsto h(a),\ y \mapsto a,\ z \mapsto h(a),\ u \mapsto a\}$	$\{u \mapsto a\}$
unifiers	$\{x \mapsto h(u),\ y \mapsto u,\ z \mapsto h(u)\}$	mgu
	$\{x \mapsto h(g(u)),\ y \mapsto g(u),\ z \mapsto h(g(u)),\ u \mapsto g(u)\}$	$\{u \mapsto g(u)\}$

Theorem

unifiable terms have mgu which can be computed by unification algorithm

Unification Algorithm

d decomposition $\dfrac{E_1,\ f(s_1,\dots,s_n) \approx f(t_1,\dots,t_n),\ E_2}{E_1,\ s_1 \approx t_1,\dots,s_n \approx t_n,\ E_2}$

t removal of trivial equations $\dfrac{E_1,\ t \approx t,\ E_2}{E_1,\ E_2}$

v variable elimination $\dfrac{E_1,\ x \approx t,\ E_2}{(E_1,\ E_2)\{x \mapsto t\}}$ and $\dfrac{E_1,\ t \approx x,\ E_2}{(E_1,\ E_2)\{x \mapsto t\}}$

if x does not occur in t (occurs check)

Example

$f(x, g(y), x) \approx f(z, g(u), h(u))$

d ⇓

$x \approx z,\ g(y) \approx g(u),\ x \approx h(u)$

v ⇓ $\{x \mapsto z\}$

$g(y) \approx g(u),\ z \approx h(u)$

d ⇓

$y \approx u,\ z \approx h(u)$

v ⇓ $\{y \mapsto u\}$

$z \approx h(u)$

v ⇓ $\{z \mapsto h(u)\}$

$\square$

mgu $\{x \mapsto h(u),\ y \mapsto u,\ z \mapsto h(u)\}$

Theorem

- there are no infinite derivations

$$U \Rightarrow_{\theta_1} V \Rightarrow_{\theta_2} W \Rightarrow_{\theta_3} \cdots$$

- if s and t are unifiable then for every maximal derivation

$$s \approx t \Rightarrow_{\theta_1} E_1 \Rightarrow_{\theta_2} E_2 \Rightarrow_{\theta_3} \cdots \Rightarrow_{\theta_n} E_n$$

  - $E_n = \square$
  - $\theta_1\theta_2\theta_3\cdots\theta_n$ is mgu of s and t

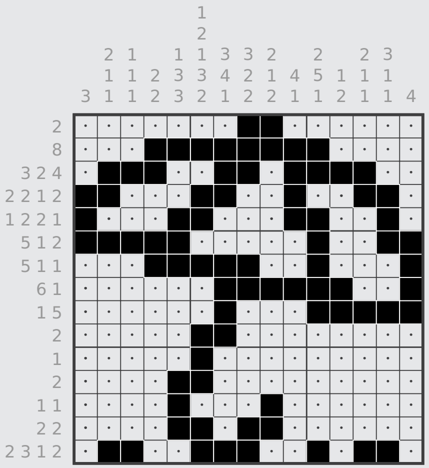
Optional Failure Rules

$\dfrac{E_1,\ f(s_1,\dots,s_n) \approx g(t_1,\dots,t_m),\ E_2}{\bot}$ $\dfrac{E_1,\ x \approx t,\ E_2}{\bot}$ $\dfrac{E_1,\ t \approx x,\ E_2}{\bot}$

if x occurs in t and $x \neq t$

Outline

Example (Picture Logic)



Outline

Definitions

- prenex normal form is predicate logic formula

$$Q_1 x_1\, Q_2 x_2 \ldots Q_n x_n\, \varphi$$

  with $Q_i \in \{\forall, \exists\}$ and φ quantifier-free

- Skolem normal form is closed (no free variables) prenex normal form

$$\forall x_1 \forall x_2 \ldots \forall x_n\, \varphi$$

  with φ quantifier-free CNF

Example

$\forall x\, \forall y\, ((P(f(x)) \vee \neg P(g(y)) \vee Q(g(y))) \wedge (\neg Q(g(y)) \vee \neg P(g(y)) \vee Q(g(x))))$

clausal form $\{\{P(f(x)), \neg P(g(y)), Q(g(y))\}, \{\neg Q(g(y)), \neg P(g(y)), Q(g(x))\}\}$

Theorem

for every formula φ there exists prenex normal form ψ such that $\varphi \equiv \psi$

Proof

① rename all bound variables such that all variables in quantifications are distinct

② push logical connectives through quantifiers:

$$\begin{array}{rr} \neg\forall x\, \varphi \equiv \exists x\, \neg\varphi & \neg\exists x\, \varphi \equiv \forall x\, \neg\varphi \\ \forall x\, \varphi \wedge \psi \equiv \forall x\, (\varphi \wedge \psi) & \varphi \wedge \forall x\, \psi \equiv \forall x\, (\varphi \wedge \psi) \\ \exists x\, \varphi \wedge \psi \equiv \exists x\, (\varphi \wedge \psi) & \varphi \wedge \exists x\, \psi \equiv \exists x\, (\varphi \wedge \psi) \\ \forall x\, \varphi \vee \psi \equiv \forall x\, (\varphi \vee \psi) & \varphi \vee \forall x\, \psi \equiv \forall x\, (\varphi \vee \psi) \\ \exists x\, \varphi \vee \psi \equiv \exists x\, (\varphi \vee \psi) & \varphi \vee \exists x\, \psi \equiv \exists x\, (\varphi \vee \psi) \\ \forall x\, \varphi \to \psi \equiv \exists x\, (\varphi \to \psi) & \varphi \to \forall x\, \psi \equiv \forall x\, (\varphi \to \psi) \\ \exists x\, \varphi \to \psi \equiv \forall x\, (\varphi \to \psi) & \varphi \to \exists x\, \psi \equiv \exists x\, (\varphi \to \psi) \end{array}$$

Theorem

for every sentence φ there exists Skolem normal form ψ such that $\varphi \approx \psi$

Proof (Skolemization)

① transform φ into closed prenex normal form $Q_1 x_1\, Q_2 x_2 \,\dots\, Q_n x_n\, \chi$ with χ in CNF

② repeatedly replace $\forall x_1 \dots \forall x_{i-1}\, \exists x_i\, Q_{i+1} x_{i+1} \dots Q_n x_n\, \psi$ by

$$\forall x_1 \dots \forall x_{i-1}\, Q_{i+1} x_{i+1} \dots Q_n x_n\, \psi[f(x_1, \dots, x_{i-1})/x_i]$$

where f is new function symbol of arity $i-1$ (if $i = 1$ then f is constant)

Remark

unification and Skolemization are required to extend resolution from propositional logic to predicate logic

Examples

❶ $\forall z\, \exists x\, \exists y\, ((P(x) \vee \neg P(y) \vee Q(z)) \wedge (\neg Q(x) \vee \neg P(y) \vee Q(z)))$

$\approx \{x \mapsto f(z)\}$

$\forall z\, \exists y\, ((P(f(z)) \vee \neg P(y) \vee Q(z)) \wedge (\neg Q(f(z)) \vee \neg P(y) \vee Q(z)))$

$\approx \{y \mapsto g(z)\}$

$\forall z\, ((P(f(z)) \vee \neg P(g(z)) \vee Q(z)) \wedge (\neg Q(f(z)) \vee \neg P(g(z)) \vee Q(z)))$

❷ $\forall x\, \exists y\, (\exists z\, P(z) \wedge (\exists u\, Q(x,u) \to \exists v\, Q(y,v)))$

$\equiv$

$\forall x\, \exists y\, \exists z\, \forall u\, \exists v\, (P(z) \wedge (\neg Q(x,u) \vee Q(y,v)))$

$\approx \{y \mapsto f(x)\}\,\{z \mapsto g(x)\}$

$\forall x\, \forall u\, \exists v\, (P(g(x)) \wedge (\neg Q(x,u) \vee Q(f(x),v)))$

$\approx \{v \mapsto h(x,u)\}$

$\forall x\, \forall u\, (P(g(x)) \wedge (\neg Q(x,u) \vee Q(f(x),h(x,u))))$

Outline

Huth and Ryan

- Section 2.3

Unification

- Wikipedia [accessed January 25, 2024]

Skolemization

- Wikipedia [accessed January 25, 2024]

Important Concepts

- at least as general
- composition
- decomposition
- instance
- most general unifier
- occurs check
- prenex normal form
- Skolem normal form
- Skolemization
- quantifier equivalences
- removal of trivial equations
- substitution
- unification algorithm
- unifier
- variable elimination

homework for May 2