universität innsbruck

SS 2024 lecture 11

Logic

Diana Gründlinger Aart Middeldorp Fabian Mitterwallner

Alexander Montag Johannes Niederhauser Daniel Rainer

Outline

Definitions

model $\mathcal{M} = (S, \rightarrow, L)$ and $X \subseteq S$

- $[\![\varphi]\!] = \{s \in S \mid \mathcal{M}, s \vDash \varphi\}$
- $\text{pre}_\forall(X) = \{s \in S \mid t \in X \text{ for all } t \text{ with } s \rightarrow t\}$
- $\text{pre}_\exists(X) = \{s \in S \mid s \rightarrow t \text{ for some } t \in X\}$

Lemma

$$[\![\top]\!] = S \qquad [\![p]\!] = \{s \in S \mid p \in L(s)\}$$

$$[\![\bot]\!] = \varnothing \qquad [\![\mathsf{AX}\,\varphi]\!] = \mathsf{pre}_\forall([\![\varphi]\!])$$

$$[\![\neg\varphi]\!] = S - [\![\varphi]\!] \qquad [\![\mathsf{EX}\,\varphi]\!] = \mathsf{pre}_\exists([\![\varphi]\!])$$

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] \qquad [\![\mathsf{AF}\,\varphi]\!] = [\![\varphi]\!] \cup \mathsf{pre}_\forall([\![\mathsf{AF}\,\varphi]\!])$$

$$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] \qquad [\![\mathsf{EF}\,\varphi]\!] = [\![\varphi]\!] \cup \mathsf{pre}_\exists([\![\mathsf{EF}\,\varphi]\!])$$

$$[\![\varphi \to \psi]\!] = (S - [\![\varphi]\!]) \cup [\![\psi]\!] \qquad [\![\mathsf{AG}\,\varphi]\!] = [\![\varphi]\!] \cap \mathsf{pre}_\forall([\![\mathsf{AG}\,\varphi]\!])$$

$$[\![\mathsf{EG}\,\varphi]\!] = [\![\varphi]\!] \cap \mathsf{pre}_\exists([\![\mathsf{EG}\,\varphi]\!])$$

$$[\![\mathsf{A}[\varphi\,\mathsf{U}\,\psi]]\!] = [\![\psi]\!] \cup ([\![\varphi]\!] \cap \mathsf{pre}_\forall([\![\mathsf{A}[\varphi\,\mathsf{U}\,\psi]]\!]))$$

$$\mathsf{pre}_\forall(X) = S - \mathsf{pre}_\exists(S - X) \qquad [\![\mathsf{E}[\varphi\,\mathsf{U}\,\psi]]\!] = [\![\psi]\!] \cup ([\![\varphi]\!] \cap \mathsf{pre}_\exists([\![\mathsf{E}[\varphi\,\mathsf{U}\,\psi]]\!]))$$

Lemma

- $[\![\mathsf{AF}\,\varphi]\!]$ is least fixed point of monotone function $F_{\mathsf{AF}}(X) = [\![\varphi]\!] \cup \mathsf{pre}_\forall(X)$
- $[\![\mathsf{EG}\,\varphi]\!]$ is greatest fixed point of monotone function $F_{\mathsf{EG}}(X) = [\![\varphi]\!] \cap \mathsf{pre}_\exists(X)$
- $[\![\mathsf{E}[\psi\,\mathsf{U}\,\varphi]]\!]$ is least fixed point of monotone function $F_{\mathsf{EU}}(X) = [\![\psi]\!] \cup ([\![\varphi]\!] \cap \mathsf{pre}_\exists(X))$

Theorem (Knaster–Tarski)

every monotone function $F\colon \mathcal{P}(S) \to \mathcal{P}(S)$ with $|S| = n$ admits

- least fixed point $\mu F = F^n(\varnothing)$
- greatest fixed point $\nu F = F^n(S)$

symbolic model checking = (CTL) model checking with BDDs

Definitions

- LTL (linear-time temporal logic) formulas are built from
  - atoms $p,\ q,\ r,\ p_1,\ p_2,\ \ldots$
  - logical connectives $\bot,\ \top,\ \neg,\ \wedge,\ \vee,\ \rightarrow$
  - temporal connectives $\mathsf{X},\ \mathsf{F},\ \mathsf{G},\ \mathsf{U},\ \mathsf{W},\ \mathsf{R}$

  according to following BNF grammar:

$$\varphi ::= \bot \mid \top \mid p \mid (\neg\varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid$$
$$(\mathsf{X}\,\varphi) \mid (\mathsf{F}\,\varphi) \mid (\mathsf{G}\,\varphi) \mid (\varphi\,\mathsf{U}\,\varphi) \mid (\varphi\,\mathsf{W}\,\varphi) \mid (\varphi\,\mathsf{R}\,\varphi)$$

- path in model $\mathcal{M} = (S, \rightarrow, L)$ is infinite sequence $s_1 \rightarrow s_2 \rightarrow \cdots$
- satisfaction $\pi \vDash \varphi$ of LTL formula φ with respect to path $\pi = s_1 \rightarrow s_2 \rightarrow \cdots$ in model $\mathcal{M}$ is defined by induction on φ
- satisfaction $\mathcal{M}, s \vDash \varphi$ of LTL formula φ with respect to state $s \in S$ in model $\mathcal{M}$ is defined as "for all paths $\pi = s \rightarrow \cdots \quad \pi \vDash \varphi$"

Definition

LTL formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if

$$\pi \vDash \varphi \quad \Longleftrightarrow \quad \pi \vDash \psi$$

for all models $\mathcal{M} = (S, \rightarrow, L)$ and paths π in $\mathcal{M}$

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Remark

$$\pi \nvDash \varphi \quad \Longleftrightarrow \quad \pi \vDash \neg\varphi$$

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Remark

$$\pi \nvDash \varphi \iff \pi \vDash \neg\varphi \qquad \mathcal{M}, s \vDash \varphi \implies \mathcal{M}, s \nvDash \neg\varphi$$

Definition

LTL formulas φ and ψ are semantically equivalent ($\varphi \equiv \psi$) if

$$\pi \vDash \varphi \quad \Longleftrightarrow \quad \pi \vDash \psi$$

for all models $\mathcal{M} = (S, \rightarrow, L)$ and paths π in $\mathcal{M}$

Remark

$$\pi \nvDash \varphi \iff \pi \vDash \neg\varphi \qquad \mathcal{M}, s \vDash \varphi \implies \mathcal{M}, s \nvDash \neg\varphi \qquad \mathcal{M}, s \nvDash \varphi \;\not\Longrightarrow\; \mathcal{M}, s \vDash \neg\varphi$$

Theorem

$$\neg \mathsf{X}\,\varphi \equiv \mathsf{X}\,\neg\varphi \qquad \varphi\,\mathsf{U}\,\psi \equiv \neg(\neg\psi\,\mathsf{U}\,(\neg\varphi \wedge \neg\psi)) \wedge \mathsf{F}\,\psi$$

$$\neg \mathsf{F}\,\varphi \equiv \mathsf{G}\,\neg\varphi \qquad \mathsf{F}\,(\varphi \vee \psi) \equiv \mathsf{F}\,\varphi \vee \mathsf{F}\,\psi$$

$$\neg \mathsf{G}\,\varphi \equiv \mathsf{F}\,\neg\varphi \qquad \mathsf{G}\,(\varphi \wedge \psi) \equiv \mathsf{G}\,\varphi \wedge \mathsf{G}\,\psi$$

$$\neg(\varphi\,\mathsf{U}\,\psi) \equiv \neg\varphi\,\mathsf{R}\,\neg\psi \qquad \mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$$

$$\neg(\varphi\,\mathsf{R}\,\psi) \equiv \neg\varphi\,\mathsf{U}\,\neg\psi \qquad \mathsf{G}\,\varphi \equiv \bot\,\mathsf{R}\,\varphi$$

$$\varphi\,\mathsf{U}\,\psi \equiv \varphi\,\mathsf{W}\,\psi \wedge \mathsf{F}\,\psi \qquad \varphi\,\mathsf{W}\,\psi \equiv \psi\,\mathsf{R}\,(\varphi \vee \psi)$$

$$\varphi\,\mathsf{W}\,\psi \equiv \varphi\,\mathsf{U}\,\psi \vee \mathsf{G}\,\varphi \qquad \varphi\,\mathsf{R}\,\psi \equiv \psi\,\mathsf{W}\,(\varphi \wedge \psi)$$

Part I: Propositional Logic

algebraic normal forms, binary decision diagrams, conjunctive normal forms, DPLL, Horn formulas, natural deduction, Post's adequacy theorem, resolution, SAT, semantics, sorting networks, soundness and completeness, syntax, Tseitin's transformation

Part II: Predicate Logic

natural deduction, quantifier equivalences, resolution, semantics, Skolemization, syntax, undecidability, unification

Part III: Model Checking

adequacy, branching-time temporal logic, CTL*, fairness, linear-time temporal logic, model checking algorithms, symbolic model checking

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Theorem

$\{X, U\}$, $\{X, W\}$ and $\{X, R\}$ are adequate sets of temporal connectives for LTL

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Proof

$$
\begin{aligned}
F\,\varphi &\equiv \top\,U\,\varphi \\
G\,\varphi &\equiv \neg F \neg \varphi \\
\varphi\,R\,\psi &\equiv \neg(\neg\varphi\,U\,\neg\psi) \\
\varphi\,W\,\psi &\equiv \varphi\,U\,\psi \vee G\,\varphi
\end{aligned}
$$

Theorem

$\{X, U\}$, $\{X, W\}$ and $\{X, R\}$ are adequate sets of temporal connectives for LTL

Proof

$$\begin{aligned} F\,\varphi &\equiv \top\,U\,\varphi \\ G\,\varphi &\equiv \neg F \neg \varphi \\ \varphi\,R\,\psi &\equiv \neg(\neg\varphi\,U\,\neg\psi) \\ \varphi\,W\,\psi &\equiv \varphi\,U\,\psi \vee G\,\varphi \end{aligned}$$

$$\begin{aligned} \varphi\,R\,\psi &\equiv \psi\,W\,(\varphi \wedge \psi) \\ \varphi\,U\,\psi &\equiv \neg(\neg\varphi\,R\,\neg\psi) \\ F\,\varphi &\equiv \top\,U\,\varphi \\ G\,\varphi &\equiv \neg F \neg \varphi \end{aligned}$$

Theorem

$\{X, U\}$, $\{X, W\}$ and $\{X, R\}$ are adequate sets of temporal connectives for LTL

Proof

$F\,\varphi \equiv \top\,U\,\varphi$	$\varphi\,R\,\psi \equiv \psi\,W\,(\varphi \wedge \psi)$	$\varphi\,U\,\psi \equiv \neg(\neg\varphi\,R\,\neg\psi)$
$G\,\varphi \equiv \neg F\,\neg\varphi$	$\varphi\,U\,\psi \equiv \neg(\neg\varphi\,R\,\neg\psi)$	$F\,\varphi \equiv \top\,U\,\varphi$
$\varphi\,R\,\psi \equiv \neg(\neg\varphi\,U\,\neg\psi)$	$F\,\varphi \equiv \top\,U\,\varphi$	$G\,\varphi \equiv \neg F\,\neg\varphi$
$\varphi\,W\,\psi \equiv \varphi\,U\,\psi \vee G\,\varphi$	$G\,\varphi \equiv \neg F\,\neg\varphi$	$\varphi\,W\,\psi \equiv \varphi\,U\,\psi \vee G\,\varphi$

Theorem

$\{\mathsf{X},\mathsf{U}\}$, $\{\mathsf{X},\mathsf{W}\}$ and $\{\mathsf{X},\mathsf{R}\}$ are adequate sets of temporal connectives for LTL

Proof

$\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$	$\varphi\,\mathsf{R}\,\psi \equiv \psi\,\mathsf{W}\,(\varphi \wedge \psi)$	$\varphi\,\mathsf{U}\,\psi \equiv \neg(\neg\varphi\,\mathsf{R}\,\neg\psi)$
$\mathsf{G}\,\varphi \equiv \neg\mathsf{F}\,\neg\varphi$	$\varphi\,\mathsf{U}\,\psi \equiv \neg(\neg\varphi\,\mathsf{R}\,\neg\psi)$	$\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$
$\varphi\,\mathsf{R}\,\psi \equiv \neg(\neg\varphi\,\mathsf{U}\,\neg\psi)$	$\mathsf{F}\,\varphi \equiv \top\,\mathsf{U}\,\varphi$	$\mathsf{G}\,\varphi \equiv \neg\mathsf{F}\,\neg\varphi$
$\varphi\,\mathsf{W}\,\psi \equiv \varphi\,\mathsf{U}\,\psi \vee \mathsf{G}\,\varphi$	$\mathsf{G}\,\varphi \equiv \neg\mathsf{F}\,\neg\varphi$	$\varphi\,\mathsf{W}\,\psi \equiv \varphi\,\mathsf{U}\,\psi \vee \mathsf{G}\,\varphi$

Theorem

$\{\mathsf{U},\mathsf{R}\}$, $\{\mathsf{U},\mathsf{W}\}$, $\{\mathsf{U},\mathsf{G}\}$, $\{\mathsf{F},\mathsf{W}\}$ and $\{\mathsf{F},\mathsf{R}\}$ are adequate sets of temporal connectives for LTL fragment consisting of negation-normal forms without X

Outline

Theorem

set of temporal connectives is adequate for CTL $\iff$

it contains $\begin{cases} \text{at least one of } \{\mathsf{AX}, \mathsf{EX}\} \\ \text{at least one of } \{\mathsf{EG}, \mathsf{AF}, \mathsf{AU}\} \\ \mathsf{EU} \end{cases}$

Theorem

set of temporal connectives is adequate for CTL $\iff$

it contains
- at least one of $\{$AX, EX$\}$
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Proof ($\Longleftarrow$)

- $\text{AX}\,\varphi \equiv \neg\,\text{EX}\,\neg\varphi$ and $\text{EX}\,\varphi \equiv \neg\,\text{AX}\,\neg\varphi$

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Proof ($\Longleftarrow$)

- $\mathsf{AX}\,\varphi \equiv \neg\,\mathsf{EX}\,\neg\varphi$ and $\mathsf{EX}\,\varphi \equiv \neg\,\mathsf{AX}\,\neg\varphi$
- $\mathsf{EF}\,\varphi \equiv \mathsf{E}[\top\,\mathsf{U}\,\varphi]$
- $\mathsf{AG}\,\varphi \equiv \neg\,\mathsf{EF}\,\neg\varphi$

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- $\mathsf{AX}\,\varphi \equiv \neg\,\mathsf{EX}\,\neg\varphi$ and $\mathsf{EX}\,\varphi \equiv \neg\,\mathsf{AX}\,\neg\varphi$
- $\mathsf{EF}\,\varphi \equiv \mathsf{E}[\top\,\mathsf{U}\,\varphi]$
- $\mathsf{AG}\,\varphi \equiv \neg\,\mathsf{EF}\,\neg\varphi$
- $\mathsf{A}[\varphi\,\mathsf{U}\,\psi] \equiv \neg(\mathsf{E}[\neg\psi\,\mathsf{U}\,(\neg\varphi \wedge \neg\psi)] \vee \mathsf{EG}\,\neg\psi)$
- $\mathsf{AF}\,\varphi \equiv \mathsf{A}[\top\,\mathsf{U}\,\varphi]$

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Proof ($\Longleftarrow$)

- $\mathsf{AX}\,\varphi \equiv \neg\,\mathsf{EX}\,\neg\varphi$ and $\mathsf{EX}\,\varphi \equiv \neg\,\mathsf{AX}\,\neg\varphi$
- $\mathsf{EF}\,\varphi \equiv \mathsf{E}[\top\,\mathsf{U}\,\varphi]$
- $\mathsf{AG}\,\varphi \equiv \neg\,\mathsf{EF}\,\neg\varphi$
- $\mathsf{AF}\,\varphi \equiv \mathsf{A}[\top\,\mathsf{U}\,\varphi]$
- $\mathsf{EG}\,\varphi \equiv \neg\,\mathsf{AF}\,\neg\varphi$

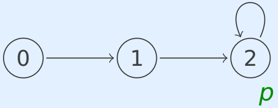
Theorem

set of temporal connectives is adequate for CTL $\iff$

it contains:
- at least one of $\{$AX, EX$\}$
- at least one of $\{$EG, AF, AU$\}$
- EU

Proof ($\implies$)

- consider model $\mathcal{M}$



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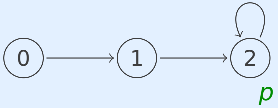

- $\mathcal{M}, 0 \not\models \mathrm{EX}\, p$ and $\mathcal{M}, 1 \models \mathrm{EX}\, p$

Theorem

set of temporal connectives is adequate for CTL $\iff$

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Proof ($\Longrightarrow$)

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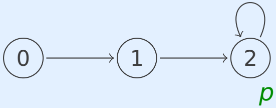
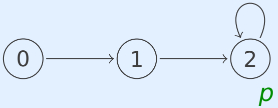


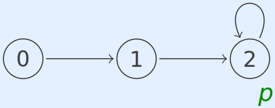
- $\mathcal{M}, 0 \nvDash \text{EX}\, p$ and $\mathcal{M}, 1 \vDash \text{EX}\, p$
- for every CTL formula φ not containing EX and AX:

$$\mathcal{M}, 0 \vDash \varphi \iff \mathcal{M}, 1 \vDash \varphi$$



Proof ($\Longrightarrow$, cont'd)

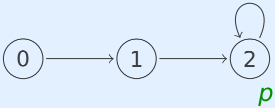
induction on φ



Proof ($\Longrightarrow$, cont'd)

induction on φ

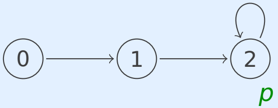
- if φ is atom or $\varphi = \bot$ then $\mathcal{M}, 0 \nvDash \varphi$ and $\mathcal{M}, 1 \nvDash \varphi$



Proof ($\Longrightarrow$, cont'd)

induction on φ

- if φ is atom or $\varphi = \bot$ then $\mathcal{M}, 0 \nvDash \varphi$ and $\mathcal{M}, 1 \nvDash \varphi$
- if $\varphi = \top$ then $\mathcal{M}, 0 \vDash \varphi$ and $\mathcal{M}, 1 \vDash \varphi$



Proof ($\Longrightarrow$, cont'd)

induction on φ

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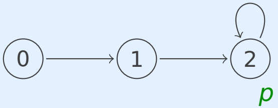
- if $\varphi = \top$ then $\mathcal{M}, 0 \vDash \varphi$ and $\mathcal{M}, 1 \vDash \varphi$
- if $\varphi = \neg\psi$ then $\mathcal{M}, 0 \vDash \varphi \iff \mathcal{M}, 0 \nvDash \psi$



Proof ($\Longrightarrow$, cont'd)

induction on φ

- if φ is atom or $\varphi = \bot$ then $\mathcal{M}, 0 \not\vDash \varphi$ and $\mathcal{M}, 1 \not\vDash \varphi$
- if $\varphi = \top$ then $\mathcal{M}, 0 \vDash \varphi$ and $\mathcal{M}, 1 \vDash \varphi$
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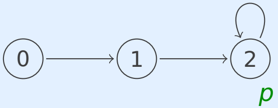

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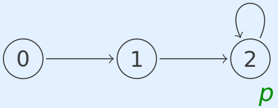

Proof ($\Longrightarrow$, cont'd)

induction on φ

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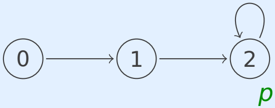
- if $\varphi = \psi_1 \wedge \psi_2$ then

$$\mathcal{M}, 0 \vDash \varphi \iff \mathcal{M}, 0 \vDash \psi_1 \text{ and } \mathcal{M}, 0 \vDash \psi_2$$



Proof ($\Longrightarrow$, cont'd)

induction on φ

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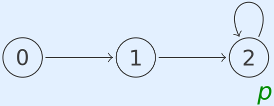
$$\begin{aligned} \mathcal{M}, 0 \vDash \varphi &\iff \mathcal{M}, 0 \vDash \psi_1 \text{ and } \mathcal{M}, 0 \vDash \psi_2 \\ &\iff \mathcal{M}, 1 \vDash \psi_1 \text{ and } \mathcal{M}, 1 \vDash \psi_2 \end{aligned}$$



Proof ($\Longrightarrow$, cont'd)

induction on φ

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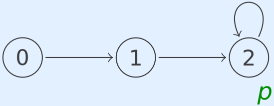
- if $\varphi = \psi_1 \wedge \psi_2$ then

$$\begin{aligned} \mathcal{M}, 0 \vDash \varphi \quad &\iff \quad \mathcal{M}, 0 \vDash \psi_1 \text{ and } \mathcal{M}, 0 \vDash \psi_2 \\ &\iff \quad \mathcal{M}, 1 \vDash \psi_1 \text{ and } \mathcal{M}, 1 \vDash \psi_2 \quad \iff \quad \mathcal{M}, 1 \vDash \varphi \end{aligned}$$

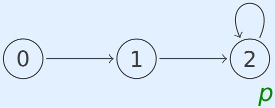


Proof ($\Longrightarrow$, cont'd)

induction on φ

- if $\varphi = \text{AF}\,\psi$ or $\varphi = \text{EF}\,\psi$ then

$$\mathcal{M}, 0 \vDash \varphi \quad \Longleftrightarrow \quad \mathcal{M}, i \vDash \psi \text{ for some } i \in \{0, 1, 2\}$$



Proof ($\Longrightarrow$, cont'd)

induction on φ

- if $\varphi = \mathsf{AF}\,\psi$ or $\varphi = \mathsf{EF}\,\psi$ then

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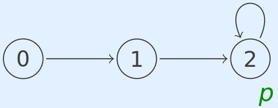

Proof ($\Longrightarrow$, cont'd)

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- if $\varphi = \mathsf{AF}\,\psi$ or $\varphi = \mathsf{EF}\,\psi$ then

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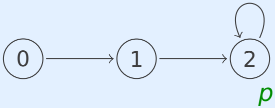

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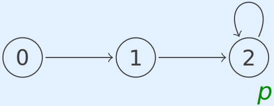

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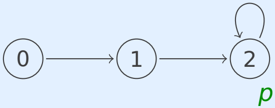

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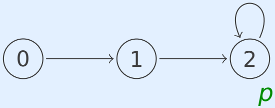

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- if $\varphi = \mathsf{A}[\psi_1 \,\mathsf{U}\, \psi_2]$ or $\varphi = \mathsf{E}[\psi_1 \,\mathsf{U}\, \psi_2]$ then

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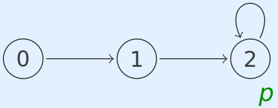

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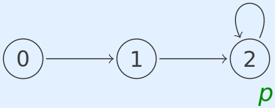

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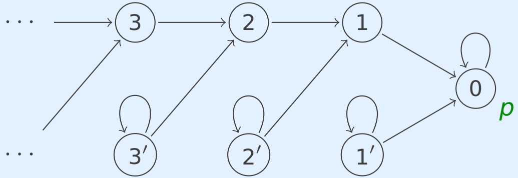
Theorem

. . . at least one of {EG, AF, AU}

Proof ($\Longrightarrow$)

- consider model $\mathcal{M}$



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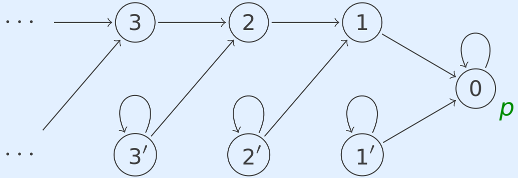

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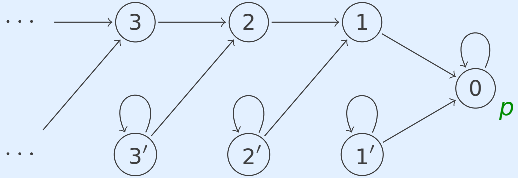

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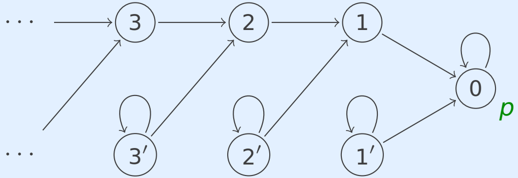

- $\mathcal{M}, i \vDash \mathrm{AF}\, p$ for all $i \geqslant 0$ and $\mathcal{M}, i' \nvDash \mathrm{AF}\, p$ for all $i > 0$
- for every CTL formula φ not containing EG, AF and AU there exists $n_\varphi > 0$ such that

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... EU

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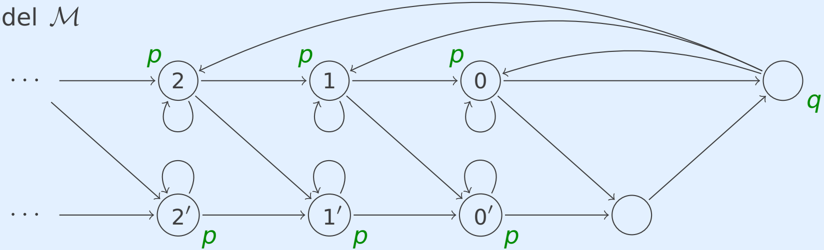

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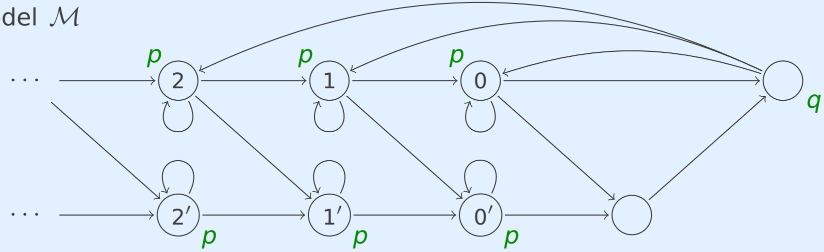

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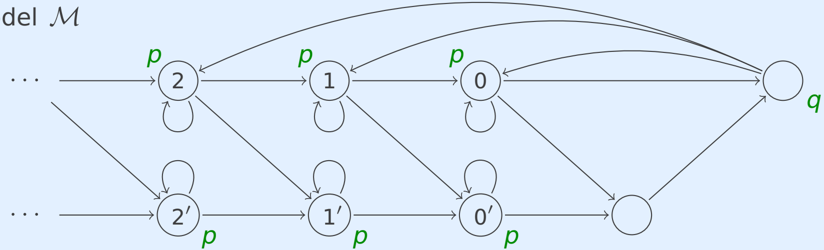

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Outline

Online Evaluation in Presence

`https://lv-analyse.uibk.ac.at/evasys/public/online/index`

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- path $s_1 \to s_2 \to \cdots$ is fair with respect to set C of CTL formulas if for all $\psi \in C$

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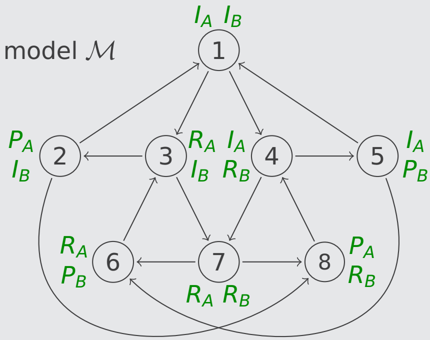
- A_C (E_C) denotes A (E) restricted to paths that are fair with respect to C

Example



- path $1\,(3\,7\,6)^\omega$ is fair with respect to $\{I_B, P_B\}$

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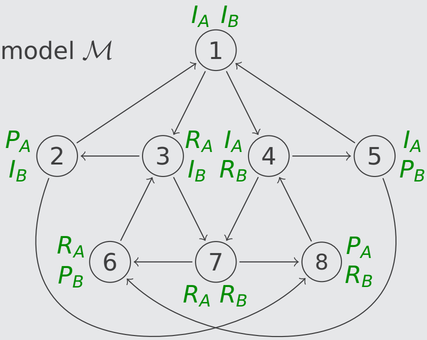

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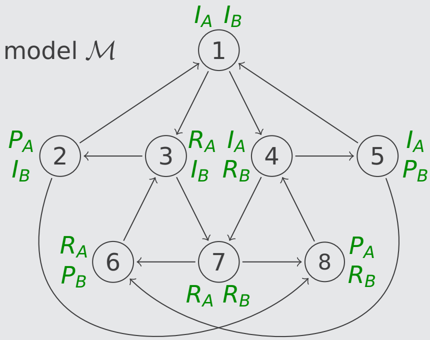

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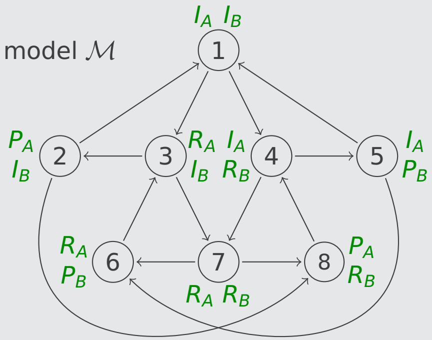

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- $\mathcal{M}, 1 \not\models \mathsf{A}_{\{R_B\}}\mathsf{F}\, P_B$ because path $1\,(4\,7\,8)^\omega$ is fair with respect to R_B but $\mathcal{M}, i \not\models P_B$ for $i \in \{1, 4, 7, 8\}$

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④ label all states in resulting SCCs

⑤ compute and label all states that can reach labelled state in restricted graph computed in step ①

Outline

1. Summary of Previous Lecture
2. Adequacy
3. Evaluation
4. Fairness
5. **Intermezzo**
6. LTL Model Checking Algorithm
7. Further Reading
8. Exam

Question

Which of the following statements hold for all models $\mathcal{M} = (S, \rightarrow, L)$ and states $s \in S$?

A $\mathcal{M}, s \models \mathsf{E}_{\{p \wedge q\}}\mathsf{F}(q)$

B $\mathcal{M}, s \not\models \mathsf{E}_{\{p\}}\mathsf{G}(\mathsf{EF}\, p)$

C $\mathcal{M}, s \models \mathsf{A}_{\{\neg q\}}\mathsf{F}(\mathsf{AX}\,\neg q)$

D $\mathcal{M}, s \models \mathsf{E}_{\{p\}}[\neg p\, \mathsf{U}\, p]$

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Example

$\mathcal{C}(a \,\mathsf{U}\, (\neg a \wedge b)) = \{a, \neg a, b, \neg b, \neg a \wedge b, \neg(\neg a \wedge b), a \,\mathsf{U}\, (\neg a \wedge b), \neg(a \,\mathsf{U}\, (\neg a \wedge b))\}$

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- $\varphi_1 \wedge \varphi_2 \in B \iff \varphi_1 \in B$ and $\varphi_2 \in B$
- $\psi \in B \implies \neg\psi \notin B$
- $\top \in \mathcal{C}(\varphi) \implies \top \in B$

② locally consistent with respect to U: for all $\varphi_1 \,\mathsf{U}\, \varphi_2 \in \mathcal{C}(\varphi)$

- $\varphi_2 \in B \implies \varphi_1 \,\mathsf{U}\, \varphi_2 \in B$
- $\varphi_1 \,\mathsf{U}\, \varphi_2 \in B$ and $\varphi_2 \notin B \implies \varphi_1 \in B$

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- $\varphi_1 \,\mathsf{U}\, \varphi_2 \in B$ and $\varphi_2 \notin B \implies \varphi_1 \in B$

③ maximal: for all $\psi \in \mathcal{C}(\varphi)$

- $\psi \notin B \implies \neg\psi \in B$

formula φ in LTL fragment with U and X as only temporal operators

Definition

closure $\mathcal{C}(\varphi)$ of φ consists of all subformulas of φ and their negations, identifying $\neg\neg\psi$ and ψ

Example

$\mathcal{C}(a\,\mathsf{U}\,(\neg a \wedge b)) = \{a, \neg a, b, \neg b, \neg a \wedge b, \neg(\neg a \wedge b), a\,\mathsf{U}\,(\neg a \wedge b), \neg(a\,\mathsf{U}\,(\neg a \wedge b))\}$

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	❶	❷	❸	❹
❶				
❷				
❸				
❹				

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	①	②	③	④
①	✓	✓		
②				
③				
④				

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	①	②	③	④
①	✓	✓		
②			✓	✓
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②			✓	✓
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	①	②	③	④
①	✓	✓		
②			✓	✓
③	✓	✓		
④			✓	✓

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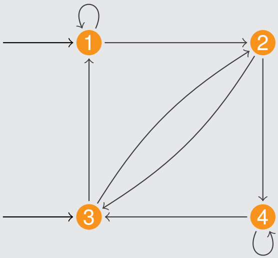
	①	②	③	④
①	✓	✓		
②			✓	✓
③	✓	✓		
④			✓	✓



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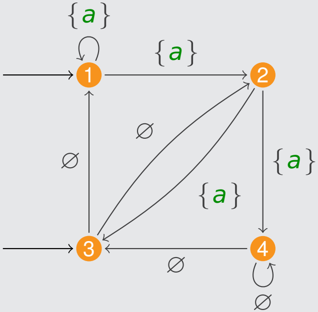
	①	②	③	④	
①	✓	✓			$\{a\}$
②			✓	✓	$\{a\}$
③	✓	✓			$\varnothing$
④			✓	✓	$\varnothing$



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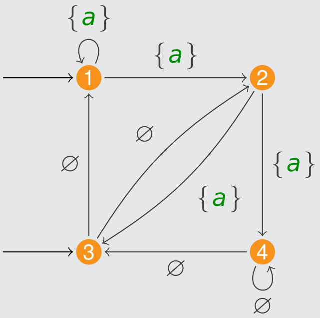

- trace $t_1 = \{a\}\{a\}\{a\}\varnothing^\omega$
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- trace t is accepted if there exists path π in A_φ such that
  1. π starts in initial state of A_φ

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  3. π visits infinitely many states satisfying $\neg(\psi_1 \,\mathsf{U}\, \psi_2) \vee \psi_2$, for every $\psi_1 \,\mathsf{U}\, \psi_2 \in \mathcal{C}(\varphi)$

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	①	②	③	④	
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②			✓	✓	$\{a\}$
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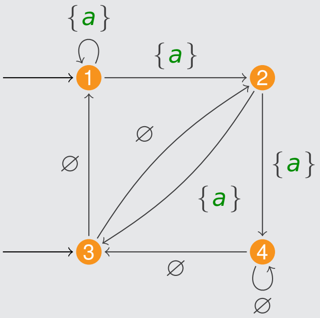

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	①	②	③	④	
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③	✓	✓			$\varnothing$
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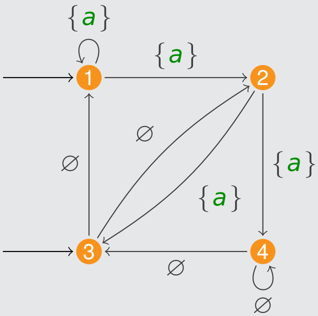

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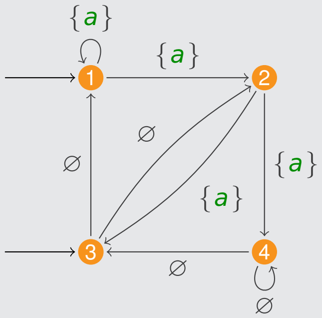
- trace $t_3 = \{a\}\varnothing\varnothing\{a\}^\omega$

Example ❶

$\varphi = \mathsf{X}\, a$

- $\mathcal{C}(\varphi) = \{a, \neg a, \mathsf{X}\, a, \neg \mathsf{X}\, a\}$
- states ① $\{a, \mathsf{X}\, a\}$ ② $\{a, \neg \mathsf{X}\, a\}$ ③ $\{\neg a, \mathsf{X}\, a\}$ ④ $\{\neg a, \neg \mathsf{X}\, a\}$
- initial states ① ③
- transitions

	①	②	③	④	
①	✓	✓			$\{a\}$
②			✓	✓	$\{a\}$
③	✓	✓			$\varnothing$
④			✓	✓	$\varnothing$



- trace $t_1 = \{a\}\{a\}\{a\}\varnothing^\omega$ is accepted: path ①①②④$^\omega$
- trace $t_2 = \varnothing\{a\}\varnothing\{a\}^\omega$ is accepted: path ③②③①$^\omega$
- trace $t_3 = \{a\}\varnothing\varnothing\{a\}^\omega$ is not accepted

Example ❷

$\varphi = a \, \mathsf{U} \, b$

Example ❷

$\varphi = a \, \mathsf{U} \, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \, \mathsf{U} \, b, \neg(a \, \mathsf{U} \, b)\}$

Example ❷

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
- states ① $\{a, b, \varphi\}$ ② $\{\neg a, b, \varphi\}$ ③ $\{a, \neg b, \varphi\}$ ④ $\{a, \neg b, \neg\varphi\}$ ⑤ $\{\neg a, \neg b, \neg\varphi\}$

Example ❷

$\varphi = a \cup b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \cup b, \neg(a \cup b)\}$
- states ① $\{a, b, \varphi\}$ ② $\{\neg a, b, \varphi\}$ ③ $\{a, \neg b, \varphi\}$ ④ $\{a, \neg b, \neg\varphi\}$ ⑤ $\{\neg a, \neg b, \neg\varphi\}$
- initial states ① ② ③

Example ❷

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
- states ① $\{a, b, \varphi\}$ ② $\{\neg a, b, \varphi\}$ ③ $\{a, \neg b, \varphi\}$ ④ $\{a, \neg b, \neg\varphi\}$ ⑤ $\{\neg a, \neg b, \neg\varphi\}$
- initial states ① ② ③
- transitions

	①	②	③	④	⑤
①	✓	✓	✓	✓	✓

$\{a, b\}$

Example ❷

$\varphi = a \, \mathsf{U} \, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \, \mathsf{U} \, b, \neg(a \, \mathsf{U} \, b)\}$
- states ① $\{a, b, \varphi\}$ ② $\{\neg a, b, \varphi\}$ ③ $\{a, \neg b, \varphi\}$ ④ $\{a, \neg b, \neg\varphi\}$ ⑤ $\{\neg a, \neg b, \neg\varphi\}$
- initial states ① ② ③
- transitions

	①	②	③	④	⑤	
①	✓	✓	✓	✓	✓	$\{a, b\}$
②	✓	✓	✓	✓	✓	$\{b\}$

Example ❷

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
- states ① $\{a, b, \varphi\}$ ② $\{\neg a, b, \varphi\}$ ③ $\{a, \neg b, \varphi\}$ ④ $\{a, \neg b, \neg\varphi\}$ ⑤ $\{\neg a, \neg b, \neg\varphi\}$
- initial states ① ② ③
- transitions

	①	②	③	④	⑤	
①	✓	✓	✓	✓	✓	$\{a, b\}$
②	✓	✓	✓	✓	✓	$\{b\}$
③	✓	✓	✓			$\{a\}$

Example ❷

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
- states ① $\{a, b, \varphi\}$ ② $\{\neg a, b, \varphi\}$ ③ $\{a, \neg b, \varphi\}$ ④ $\{a, \neg b, \neg\varphi\}$ ⑤ $\{\neg a, \neg b, \neg\varphi\}$
- initial states ① ② ③
- transitions

	①	②	③	④	⑤	
①	✓	✓	✓	✓	✓	$\{a, b\}$
②	✓	✓	✓	✓	✓	$\{b\}$
③	✓	✓	✓			$\{a\}$
④				✓	✓	$\{a\}$

Example ❷

$\varphi = a \, \mathsf{U} \, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \, \mathsf{U} \, b, \neg(a \, \mathsf{U} \, b)\}$
- states ❶ $\{a, b, \varphi\}$ ❷ $\{\neg a, b, \varphi\}$ ❸ $\{a, \neg b, \varphi\}$ ❹ $\{a, \neg b, \neg\varphi\}$ ❺ $\{\neg a, \neg b, \neg\varphi\}$
- initial states ❶ ❷ ❸
- transitions

	❶	❷	❸	❹	❺	
❶	✓	✓	✓	✓	✓	$\{a, b\}$
❷	✓	✓	✓	✓	✓	$\{b\}$
❸	✓	✓	✓			$\{a\}$
❹				✓	✓	$\{a\}$
❺	✓	✓	✓	✓	✓	$\varnothing$

Example ②

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
- states ① $\{a, b, \varphi\}$ ② $\{\neg a, b, \varphi\}$ ③ $\{a, \neg b, \varphi\}$ ④ $\{a, \neg b, \neg\varphi\}$ ⑤ $\{\neg a, \neg b, \neg\varphi\}$
- initial states ① ② ③
- transitions

	①	②	③	④	⑤	
①	✓	✓	✓	✓	✓	$\{a, b\}$
②	✓	✓	✓	✓	✓	$\{b\}$
③	✓	✓	✓			$\{a\}$
④				✓	✓	$\{a\}$
⑤	✓	✓	✓	✓	✓	$\varnothing$

- acceptance condition : paths cycling in state ③ are not accepting

Example ❷

$\varphi = a \,\mathsf{U}\, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \,\mathsf{U}\, b, \neg(a \,\mathsf{U}\, b)\}$
- states ① $\{a, b, \varphi\}$ ② $\{\neg a, b, \varphi\}$ ③ $\{a, \neg b, \varphi\}$ ④ $\{a, \neg b, \neg\varphi\}$ ⑤ $\{\neg a, \neg b, \neg\varphi\}$
- initial states ① ② ③
- transitions

	①	②	③	④	⑤	
①	✓	✓	✓	✓	✓	$\{a, b\}$
②	✓	✓	✓	✓	✓	$\{b\}$
③	✓	✓	✓			$\{a\}$
④				✓	✓	$\{a\}$
⑤	✓	✓	✓	✓	✓	$\varnothing$

- acceptance condition: paths cycling in state ③ are not accepting
- $\{a\}^\omega$ is rejected

Example ❷

$\varphi = a \, \mathsf{U} \, b$

- $\mathcal{C}(\varphi) = \{a, \neg a, b, \neg b, a \, \mathsf{U} \, b, \neg(a \, \mathsf{U} \, b)\}$
- states ① $\{a, b, \varphi\}$ ② $\{\neg a, b, \varphi\}$ ③ $\{a, \neg b, \varphi\}$ ④ $\{a, \neg b, \neg\varphi\}$ ⑤ $\{\neg a, \neg b, \neg\varphi\}$
- initial states ① ② ③
- transitions

	①	②	③	④	⑤	
①	✓	✓	✓	✓	✓	$\{a, b\}$
②	✓	✓	✓	✓	✓	$\{b\}$
③	✓	✓	✓			$\{a\}$
④				✓	✓	$\{a\}$
⑤	✓	✓	✓	✓	✓	$\varnothing$

- acceptance condition: paths cycling in state ③ are not accepting
- $\{a\}^\omega$ is rejected and $\{b\} \varnothing \{a\}^\omega$ is accepted

Basic Strategy

$\mathcal{M}, s \vDash \varphi$?

- construct labelled Büchi automaton $A_{\neg\varphi}$ for $\neg\varphi$
- combine $A_{\neg\varphi}$ and $\mathcal{M}$ into single automaton $A_{\neg\varphi} \times \mathcal{M}$
- determine whether there exists accepting path in $A_{\neg\varphi} \times \mathcal{M}$

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

		1	2	3	4	5
	$\{a, b, \varphi\}$ 1	✓	✓	✓	✓	✓
	$\{\neg a, b, \varphi\}$ 2	✓	✓	✓	✓	✓
	$\{a, \neg b, \varphi\}$ 3	✓	✓	✓		
$\rightarrow$	$\{a, \neg b, \neg\varphi\}$ 4				✓	✓
$\rightarrow$	$\{\neg a, \neg b, \neg\varphi\}$ 5	✓	✓	✓	✓	✓

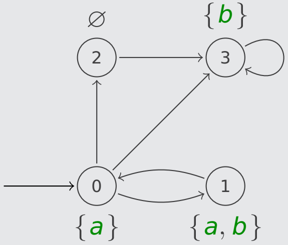
acceptance condition: paths cycling in state 3 are not accepting

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathrm{U}\, b$

		1	2	3	4	5
	$\{a, b, \varphi\}$ 1	✓	✓	✓	✓	✓
	$\{\neg a, b, \varphi\}$ 2	✓	✓	✓	✓	✓
	$\{a, \neg b, \varphi\}$ 3	✓	✓	✓		
$\rightarrow$	$\{a, \neg b, \neg\varphi\}$ 4				✓	✓
$\rightarrow$	$\{\neg a, \neg b, \neg\varphi\}$ 5	✓	✓	✓	✓	✓

model $\mathcal{M}$



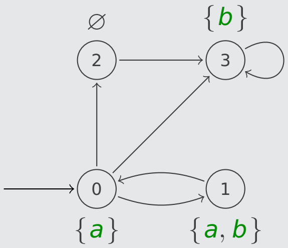
acceptance condition: paths cycling in state 3 are not accepting

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	1	2	3	4	5
$\{a, b, \varphi\}$ 1	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ 2	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ 3	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ 4				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ 5	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state 3 are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

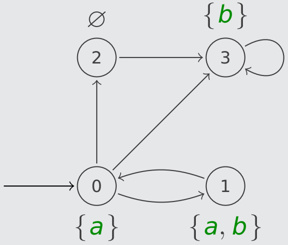
$\rightarrow$ 4 0	
$\rightarrow$ 5 0	

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	①	②	③	④	⑤
$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

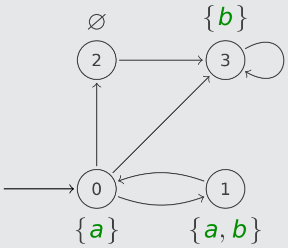
$\rightarrow$ ④ 0	{ ④ 1, ④ 2, ④ 3, ⑤ 1, ⑤ 2, ⑤ 3 }
$\rightarrow$ ⑤ 0	

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	①	②	③	④	⑤
$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

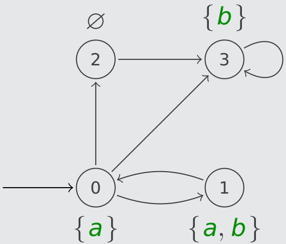
$\rightarrow$ ④ 0	{④ 1, ④ 2, ④ 3, ⑤ 1, ⑤ 2, ⑤ 3}
$\rightarrow$ ⑤ 0	$\varnothing$

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

		①	②	③	④	⑤
	$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
	$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
	$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\rightarrow$	$\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\rightarrow$	$\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

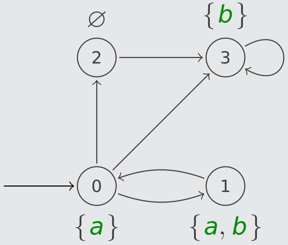
$\rightarrow$ ④0	{ ④2, ④3, ⑤1, ⑤2, ⑤3 }
$\rightarrow$ ⑤0	$\varnothing$

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

		①	②	③	④	⑤
	$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
	$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
	$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\to$	$\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\to$	$\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

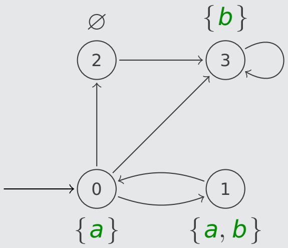
$\to$ ④0	{ ④3, ⑤1, ⑤2, ⑤3 }
$\to$ ⑤0	$\varnothing$

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	①	②	③	④	⑤
$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

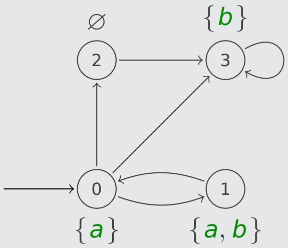
$\rightarrow$ ④ 0	{ ⑤ 1, ⑤ 2, ⑤ 3 }
$\rightarrow$ ⑤ 0	$\varnothing$

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

		1	2	3	4	5
	$\{a, b, \varphi\}$ 1	✓	✓	✓	✓	✓
	$\{\neg a, b, \varphi\}$ 2	✓	✓	✓	✓	✓
	$\{a, \neg b, \varphi\}$ 3	✓	✓	✓		
$\rightarrow$	$\{a, \neg b, \neg\varphi\}$ 4				✓	✓
$\rightarrow$	$\{\neg a, \neg b, \neg\varphi\}$ 5	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state 3 are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

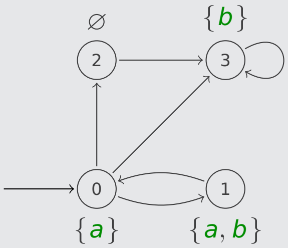
$\rightarrow$ 4 0	$\{$	5 2, 5 3 $\}$
$\rightarrow$ 5 0	$\varnothing$	

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	①	②	③	④	⑤
$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

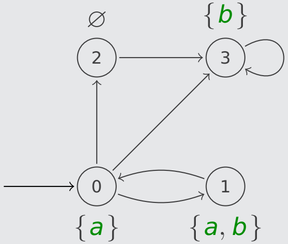
$\rightarrow$ ④ 0	{ ⑤ 2 }
$\rightarrow$ ⑤ 0	$\varnothing$

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

		①	②	③	④	⑤
	$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
	$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
	$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\to$	$\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\to$	$\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

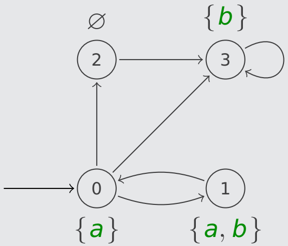
$\to$ ④0	$\{$ ⑤2 $\}$	
$\to$ ⑤0	$\varnothing$	
⑤2	$\{$①3, ②3, ③3, ④3, ⑤3$\}$	

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \, \mathsf{U} \, b$

		1	2	3	4	5
	$\{a, b, \varphi\}$ 1	✓	✓	✓	✓	✓
	$\{\neg a, b, \varphi\}$ 2	✓	✓	✓	✓	✓
	$\{a, \neg b, \varphi\}$ 3	✓	✓	✓		
$\rightarrow$	$\{a, \neg b, \neg\varphi\}$ 4				✓	✓
$\rightarrow$	$\{\neg a, \neg b, \neg\varphi\}$ 5	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state 3 are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

$\rightarrow$ 4 0 | { 5 2 }   5 2 | { 2 3 }

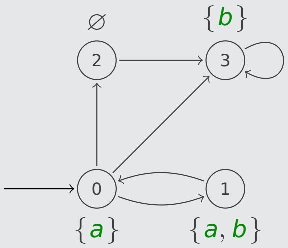
$\rightarrow$ 5 0 | ∅

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	①	②	③	④	⑤
$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

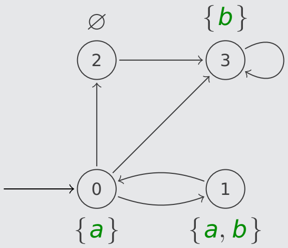
$\rightarrow$ ④ 0	{ ⑤ 2 }	⑤ 2	{ ② 3 }
$\rightarrow$ ⑤ 0	$\varnothing$	② 3	{① 3, ② 3, ③ 3, ④ 3, ⑤ 3}

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	①	②	③	④	⑤
$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\to$ $\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\to$ $\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

$\to$ ④ 0 | { ⑤ 2 }

$\to$ ⑤ 0 | $\varnothing$

⑤ 2 | { ② 3 }

② 3 | { ② 3 }

Basic Strategy

$\mathcal{M}, s \models \varphi$?

- construct labelled Büchi automaton $A_{\neg\varphi}$ for $\neg\varphi$
- combine $A_{\neg\varphi}$ and $\mathcal{M}$ into single automaton $A_{\neg\varphi} \times \mathcal{M}$
- determine whether there exists accepting path in $A_{\neg\varphi} \times \mathcal{M}$

Theorem

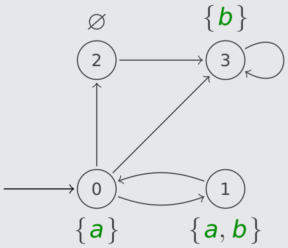
$\mathcal{M}, s \models \varphi \quad \iff \quad A_{\neg\varphi} \times \mathcal{M}$ has no accepting paths

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \,\mathsf{U}\, b$

	①	②	③	④	⑤
$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

$\rightarrow$ ④ 0	{ ⑤ 2 }	⑤ 2	{ ② 3 }
$\rightarrow$ ⑤ 0	$\varnothing$	② 3	{ ② 3 }

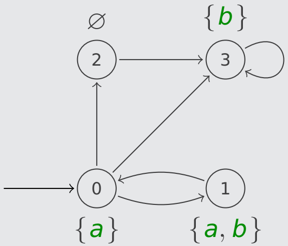
- accepting path ④ 0 $\xrightarrow{\{a\}}$ ⑤ 2 $\xrightarrow{\varnothing}$ ② 3 $\xrightarrow{\{b\}}$ ② 3 $\xrightarrow{\{b\}}$ $\cdots$

Example

labelled Büchi automaton $A_{\neg\varphi}$ for $\varphi = a \, \mathsf{U} \, b$

	①	②	③	④	⑤
$\{a, b, \varphi\}$ ①	✓	✓	✓	✓	✓
$\{\neg a, b, \varphi\}$ ②	✓	✓	✓	✓	✓
$\{a, \neg b, \varphi\}$ ③	✓	✓	✓		
$\rightarrow$ $\{a, \neg b, \neg\varphi\}$ ④				✓	✓
$\rightarrow$ $\{\neg a, \neg b, \neg\varphi\}$ ⑤	✓	✓	✓	✓	✓

model $\mathcal{M}$



acceptance condition: paths cycling in state ③ are not accepting

- product automaton $A_{\neg\varphi} \times \mathcal{M}$

$\rightarrow$ ④0 | { ⑤2 } ⑤2 | { ②3 }

$\rightarrow$ ⑤0 | ∅ ②3 | { ②3 }

- accepting path ④0 $\xrightarrow{\{a\}}$ ⑤2 $\xrightarrow{\varnothing}$ ②3 $\xrightarrow{\{b\}}$ ②3 $\xrightarrow{\{b\}}$ $\cdots$ $\Longrightarrow$ $\mathcal{M}, 0 \nvDash \varphi$

Outline

Huth and Ryan

- Section 3.2.5
- Section 3.4.5
- Section 3.6.2
- Section 3.6.3

Huth and Ryan

- Section 3.2.5
- Section 3.4.5
- Section 3.6.2
- Section 3.6.3

Baier and Katoen

- Section 5.2 of Principles of Model Checking (MIT Press 2008)

Important Concepts

- A_C
- A_φ
- adequacy
- closure
- E_C
- elementary set
- fair path
- fairness constraints
- labelled Büchi automaton
- trace

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homework for June 6

Outline

First Exam on June 24

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- visit consultation hours AM Wednesday, 11:30 – 13:00, 3M07