Sheet 6 Deadline: May 7, 2024, 3pm

- Prepare your solutions on paper.
- Mark the exercises in OLAT before the deadline.
- Marking an exercise means that a significant part of that exercise has been treated.

Exercise 1 *Strong Normalization of Lexicographic Combinations* **7 p.**

1. On slide 4/40, it was argued that termination holds because of a lexicographic measure. In this exercise we want to be a bit more formal about this aspect by showing that taking lexicographic combinations is a valid technique for termination proving.

   Given n binary relations $\succ_1, \dots, \succ_n$ over sets $A_1, \dots, A_n$, we define their lexicographic combination $\succ_{lex}$ as a binary relation over $A_1 \times \ldots \times A_n$ as follows: a lexicographic decrease happens, if for some position i, the element at position i decreases w.r.t. $\succ_i$, the elements before position i are unchanged, and there is no restriction on the elements after position i. This can be made formal via the following inference rule:

   $$\frac{1 \le i \le n \quad a_i \succ_i b_i}{(a_1, \dots, a_{i-1}, a_i, \dots, a_n) \succ_{lex} (a_1, \dots, a_{i-1}, b_i, \dots, b_n)}$$

   Prove that whenever $SN(\succ_i)$ for all $1 \le i \le n$, then also $SN(\succ_{lex})$ is satisfied. (4 points)

2. Find a lexicographic combination of strongly normalizing relations such that all rules of the pattern completeness algorithm on slide 4/44 decrease w.r.t. that combination. You can inline and merge rules as on slide 4/47, i.e., you only need to consider the (inlined) rules of (decompose), (match), (clash, merged with remove-mp), (success, merged with remove-pp), (failure), and (instantiate).

   Here, you may assume that $\succ_1$ is defined as the relation on slide 4/51 such that $P \succ_1 P'$ whenever P is instantiated to P', and $P \succeq_1 P'$ for all other (inlined) rules. (3 points)

Exercise 2 *Termination Analysis on Paper* **4 p.**

Write your favourite sorting algorithm as functional program and try to prove termination via the subterm criterion and size-change termination. If the proof is not completed, indicate which dependency pairs remain. Of course, here you also have to define a function for comparing natural numbers and other auxiliary functions. But you can assume that there are already datatypes Nat, List and Bool.

Exercise 3 *Size-Change Termination* **9 p.**

In the lecture the set of multigraphs $\mathcal{M}$ of a set of size-change graphs $\mathcal{G}$ has essentially been defined as follows:

$$\frac{G \in \mathcal{G}}{G \in \mathcal{M}} \qquad\qquad \frac{G_1 \in \mathcal{M} \quad G_2 \in \mathcal{M}}{G_1 \bullet G_2 \in \mathcal{M}}$$

Now consider the following set of multigraphs $\mathcal{N}$, defined as:

$$\frac{G \in \mathcal{G}}{G \in \mathcal{N}} \qquad\qquad \frac{G_1 \in \mathcal{G} \quad G_2 \in \mathcal{N}}{G_1 \bullet G_2 \in \mathcal{N}}$$

In this exercise we will show that both definitions are equivalent and also compare the subterm criterion with size-change termination.

1. Think about the relation-ship between the subterm criterion and size-change termination. Does one criterion subsume the other one? For both directions, either give a proof of the subsumption or provide a concrete counter example. (2 points)
2. Prove $\mathcal{N} \subseteq \mathcal{M}$. (2 points)
3. Prove $\mathcal{M} \subseteq \mathcal{N}$. You can assume that $\bullet$ is associative. Most likely, you will need to prove one auxiliary property. (4 points)
4. Think about an implementation: is it faster to compute $\mathcal{M}$ or $\mathcal{N}$? Why? (1 point)