universität
innsbruck

Program Verification	SS 2024	LVA 703083+703084
Sheet 7		Deadline: May 14, 2024, 3pm

- Prepare your solutions on paper.
- Mark the exercises in OLAT before the deadline.
- Marking an exercise means that a significant part of that exercise has been treated.

Exercise 1 *Ramsey's Theorem* **3 p.**

Assume our universe has infinitely many planets, where on these planets infinitely many different species are living, e.g., humans, bees, cats, dogs, flies, marsians, vulcans, marsupilamis, cremadingdongs, ...
Prove the following result with help of Ramsey's theorem. There is an infinite set S of species, such that either all species within S can communicate with all other species of S, or no species in S can communicate with any of the other species in S.
In your argumentation clarify how you choose the parameters X, c, n, C within Ramsey's theorem.

Exercise 2 *Usable Equations and Polynomial Interpretations* **17 p.**

The usable equations can be refined to usable equations w.r.t. an argument filter where an argument filter describes for each function symbol which arguments of that symbol are considered and which are ignored by the reduction pair.
Formally, an argument filter is a map π such that $\pi(F) \subseteq \{1, \dots, n\}$ for every n-ary function symbol F.
The idea is now to refine usable equations in a way that only those defined symbols are considered that appear on positions in right-hand sides which are not ignored.
For instance, for the set dependency pairs P that consists only of

$$\mathsf{f}^\sharp(\mathsf{Cons}(x, xs), \mathsf{Succ}(y)) \to \mathsf{f}^\sharp(\mathsf{reverse}(xs), \mathsf{g}(x, y)) \tag{1}$$

we might use $\pi(\mathsf{f}^\sharp) = \{1\}$ and then only get usable equations for reversal, but not for function g. This should lead to a successful termination proof since the first argument gets shorter, and it does not matter what is happening in the second argument, i.e., it does not matter how complex the g-equations are.

1. Provide a definition of usable equations w.r.t. a fixed argument filter π. In particular, there should be a definition of $\mathcal{U}(t)$, the usable equations of a term, where t is a (subterm) of a right-hand side. You can assume that $\mathcal{E}_f$ is the set of equations of the program that defines f. (3 points)
2. Provide a definition that a reduction pair $(\succ, \succsim)$ is compatible with an argument filter, i.e., that whenever $i \notin \pi(F)$, then the i-th argument of F must be ignored by $\succsim$. (1 point)
3. Prove soundness of usable rules: whenever $\ell \succsim r$ for all usable equations of $\ell = r \in \mathcal{U}(t)$ and $t\sigma \overset{i}{\hookrightarrow}{}^* s$ for some normal-form substitution σ, then $t\sigma \succsim s$. W.l.o.g. you may assume that $\succsim$ is reflexive. (5 points)
4. One problem in the automation of usable equations w.r.t. and argument filter is that we do not want to fix π in advance, i.e., we want to avoid having to iterate over all possible choices of π. Therefore, one usually also treats π symbolically, i.e., "$i \in \pi(F)$" will be seen as a Boolean variable in the SMT-encoding for polynomials. Similarly, one adds Boolean variables u_f to indicate that the f-equations are usable.

   Using these ideas, design an encoding to search for linear polynomial interpretations and apply it on the example set of dependency pairs P from above and the following equations. You can just write $encode(n)$

for the SMT-encoding of $\ell \succsim r$ (or $s \succ t$) where n corresponds to the number of the equation $\ell = r$ (or of dependency pair $s \to t$). (5 points)

$$\begin{aligned}
\mathsf{reverse}(xs) &= \mathsf{rev}(xs, \mathsf{Nil}) && (2)\\
\mathsf{rev}(\mathsf{Nil}, ys) &= ys && (3)\\
\mathsf{rev}(\mathsf{Cons}(x, xs), ys) &= \mathsf{rev}(xs, \mathsf{Cons}(x, ys)) && (4)\\
\mathsf{g}(x, y) &= \mathsf{double}(\mathsf{incr}(x)) && (5)\\
\mathsf{double}(\mathsf{Zero}) &= \mathsf{Zero} && (6)\\
\mathsf{double}(\mathsf{Succ}(x)) &= \mathsf{Succ}(\mathsf{Succ}(\mathsf{double}(x))) && (7)\\
\mathsf{incr}(x) &= \mathsf{Succ}(x) && (8)
\end{aligned}$$

5. Provide a polynomial interpretation that solves the constraints. Hint: choosing $\pi(\mathsf{f}^\sharp) = \{1\}$ is a good idea, and of course, the strict decrease happens because of a decrease in list-length.

   If you did not solve the parts on usable equations, just try to find a polynomial interpretation that satisfies those constraints that we get for P and for equations (2) – (4). (3 points)