- Prepare your solutions on paper.
- Mark the exercises in OLAT before the deadline.
- Marking an exercise means that a significant part of that exercise has been treated.

Exercise 1 *Axioms about Equality* — 13 p.

On slide 5/25 we have seen that there are not enough properties about the equality predicates $=_\tau$. Among these missing properties are reflexivity and symmetry of $=_\tau$.
One way to get access to these properties is by proving that these formulas are valid in the standard model.

1. Prove that $=_\tau$ is symmetric, i.e., $\mathcal{M} \models \forall x, y.\, x =_\tau y \longleftrightarrow y =_\tau x$ for the standard model $\mathcal{M}$. (2 points)

Instead of proving these properties via the standard model, an alternative approach is to show that the missing properties are already consequences of AX (assuming that AX includes the induction formulas as well as the decomposition- and disjointness-theorems of slide 5/11).

2. Show $AX \models \forall xs.\, xs =_{\mathsf{List}} xs$ via a natural deduction proof in the style of slide 5/25.

   Here, you can assume the standard definition of lists of natural numbers via Nil and Cons and you can assume $\forall x.\, x =_{\mathsf{Nat}} x$ as axiom (which can be proven in a similar way). For the reasoning about Boolean connectives you can be sloppy, but you should be detailed in proof steps that involve using axioms of AX, i.e., make precise which axioms you are using. (4 points)

3. Show that $=_{\mathsf{Nat}}$ is symmetric by deducing $AX \models \forall x, y.\, x =_{\mathsf{Nat}} y \longrightarrow y =_{\mathsf{Nat}} x$ as in part (2).
   Hint: Use induction on x for the formula $\psi := \forall y.\, x =_{\mathsf{Nat}} y \longrightarrow y =_{\mathsf{Nat}} x$.

   Be careful with the correct usage of quantifiers in your proof.

   (a) Write down the fully spelled out formula that you get when using ψ in the induction scheme for natural numbers. (2 points)

   (b) Perform the proof. For this part you can assume an additional axiom scheme for case analysis on natural numbers:

   $$\vec{\forall}\,(\varphi[x/\mathsf{Zero}] \longrightarrow (\forall z.\, \varphi[x/\mathsf{Succ}(z)]) \longrightarrow \varphi)$$

   where z must be fresh for φ.
   Hint: in each case of the induction proof, you need to perform one case analysis. (5 points)

Exercise 2 *Induction Formulas* — 7 p.

1. Consider the datatype

   $$\textsf{data Tree} = \mathsf{Empty} : \mathsf{Tree} \mid \mathsf{Node} : \mathsf{Tree} \times \mathsf{Nat} \times \mathsf{Tree} \to \mathsf{Tree}$$

   and that there are two functions on trees

   $$\mathsf{flatten} : \mathsf{Tree} \to \mathsf{List}$$
   $$\mathsf{mergeTree} : \mathsf{Tree} \times \mathsf{Tree} \to \mathsf{Tree}$$

and there is the standard append function on lists.

One might want to prove the following property by induction on t_1:

$$\forall t_1, t_2.\, \mathsf{flatten}(\mathsf{mergeTree}(t_1, t_2)) =_{\mathsf{List}} \mathsf{append}(\mathsf{flatten}(t_1), \mathsf{flatten}(t_2)) \tag{1}$$

In this exercise you should just write down the induction formula that you get for proving (1). To this end, first write down the generic induction formula for datatype Tree and an arbitrary formula φ, cf. slide 5/22. Next, write down the induction formula that you get when instantiating φ according to (1) and then apply all substitutions; don't forget to mention conditions on freshness of variables. (3 points)

2. Soundness of the induction formulas has been proven on slide slide 5/23. However, the proof never explicitly mentions freshness of the variables, which is essential (cf. slide 5/21).

   Study the proof on slide slide 5/23 and extend the argumentation. In particular it should be indicated where the freshness condition is essential. (4 points)