- Prepare your solutions on paper.
- Mark the exercises in OLAT before the deadline.
- If you use Isabelle for Exercise 2, then upload your Isabelle file in OLAT.
- Marking an exercise means that a significant part of that exercise has been treated.

Exercise 1 *Contexts and Subterms* **9 p.**

The set of contexts $\mathcal{C}$ can formally be defined as follows, where $\Box$ represents the hole and types are ignored for simplicity.

$$\frac{}{\Box \in \mathcal{C}} \qquad \frac{t_1 \in \mathcal{T}(\Sigma, \mathcal{V}) \quad \dots \quad C \in \mathcal{C} \quad \dots \quad t_n \in \mathcal{T}(\Sigma, \mathcal{V})}{f(t_1, \dots, C, \dots, t_n)}$$

1. Provide a recursive definition of plugging a term t into the hole of a context C, i.e., define $C[t]$. (1 point)
2. Prove $s \hookrightarrow t \longrightarrow C[s] \hookrightarrow C[t]$. Which induction scheme are you using? (2 points)
3. Prove $s \trianglerighteq t \longrightarrow \exists C.\ s = C[t]$. Which induction scheme are you using? (2 points)
4. Prove that $SN(\hookrightarrow)$ implies $SN(\hookrightarrow \circ \trianglerighteq)$ where $\circ$ here denotes relation composition. (4 points)
   Hint: You should start your proof with an infinite $(\hookrightarrow \circ \trianglerighteq)$-sequence and then show that from this sequence one can construct an infinite $\hookrightarrow$-sequence with the help of the previous results.

   Note that this is the main property to ensure that the induction rule wrt. algorithms (slide 5/56) is an instance of well-founded induction. The reason is that there we provide IHs which correspond to subterms of rhss. Therefore, for each equation $\ell = r$ and subterm u of r that leads to an IH, we have $\ell \hookrightarrow r \trianglerighteq u$, so the subterm u of the rhs is smaller than the lhs ℓ wrt. the strongly normalizing relation $\hookrightarrow \circ \trianglerighteq$.

Exercise 2 *Equational Reasoning and Induction* **11 p.**

Consider the following program.

$$\begin{aligned}
&\mathsf{plus}(\mathsf{Succ}(x), y) = \mathsf{Succ}(\mathsf{plus}(x, y))\\
&\mathsf{plus}(\mathsf{Zero}, y) = y\\
&\mathsf{times}(\mathsf{Succ}(x), y) = \mathsf{plus}(y, \mathsf{times}(x, y))\\
&\mathsf{times}(\mathsf{Zero}, y) = \mathsf{Zero}\\
&\mathsf{sum_nums}(\mathsf{Zero}) = \mathsf{Zero}\\
&\mathsf{sum_nums}(\mathsf{Succ}(x)) = \mathsf{plus}(\mathsf{Succ}(x), \mathsf{sum_nums}(x))
\end{aligned}$$

1. Prove commutativity of multiplication, i.e.,

   $$\forall x, y.\ \mathsf{times}(x, y) =_{\mathsf{Nat}} \mathsf{times}(y, x).$$

   To solve this problem you will need to identify and prove several additional properties. You can assume that plus is commutative, as this property was already proven in the lecture. (6 points)

2. Prove the Gaussian formula, i.e.,

$$\forall x.\ \mathsf{times}(\mathsf{sum\_nums}(x), \mathsf{Succ}(\mathsf{Succ}(\mathsf{Zero}))) =_{\mathsf{Nat}} \mathsf{times}(x, \mathsf{Succ}(x)).$$

Also for this problem you will need to identify and prove some additional properties. (5 points)

You can perform the proofs on paper or by using Isabelle.

https://isabelle.in.tum.de

In the former case, you do NOT have to specify every singleton simplification step, but just give some intermediate results in the simplification process.
In the latter case, a template file is available on the website. This contains some setup and guidelines on how to use Isabelle.